LECTURE 9

The Riemann Integral: Construction, Integrability, and the Fundamental Theorem

MATH S4061 | Introduction to Modern Analysis I

Thursday, 25 June 2026

I take the bills and coins out of my pocket and give them to the creditor in the order I find them … This is the Riemann integral. But I can proceed differently: … I order the bills and coins according to identical values and … pay the several heaps one after the other … This is my integral.

Henri Lebesgue, letter to Paul Montel

Contents of this lecture

Overview. The calculus slogan is "area under the curve," but here the integral comes first: area is *defined by* the integral, not the reverse. We build the Riemann integral for any *bounded* f on a closed bounded interval—continuity is not required—by trapping the curve between under- and over-estimating staircases, and we call f integrable exactly when the two estimates can be made to agree. Refining a partition squeezes the sums together, and that single fact yields the working test for integrability: $U - L$ can be made smaller than any ε. The test then delivers three classes of integrable functions—continuous, monotone, and boundedly discontinuous at finitely many points—and points past Ch. 6 to the full Lebesgue criterion, which the Cantor set shows is genuinely sharper. We close by recording the algebraic properties of the integral and the two halves of the Fundamental Theorem, whose proofs open the next lecture. The companion text is Rudin, Chapter 6, read throughout with $\alpha(x) = x$, so $\int f\,d\alpha$ becomes $\int f\,dx$ (R 6.1–R 6.21).

9.1 Upper and Lower Integrals

In the calculus picture one partitions $[a, b]$ into equal pieces, samples f at a point x_i of each, and reads $\int_a^b f(x)\,dx$ as the limit of the Riemann sums $R_n(f) = \sum_{i=1}^n f(x_i)\,\Delta x_i$. We will be more general: continuity is *not* required, and many discontinuous functions still turn out to be integrable. We ask only that

$$f : [a, b] \to \mathbb{R} \quad \text{be a } \textit{bounded} \text{ function}, \qquad M := \sup_{[a,b]} |f(x)| < \infty,$$

on a closed and bounded interval.[1]

Definition 9.1.1

Partition of $[a, b]$

A *partition* of $[a, b]$ is a finite set $P = \{x_0, x_1, \ldots, x_n\} \subseteq [a, b]$ with

$$a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b,$$

and we write $\Delta x_i = x_i - x_{i-1}$ for the length of the i-th subinterval (R 6.1). Consecutive points may coincide.

Definition 9.1.2

Upper and lower sums

Given a partition P of $[a, b]$, set

$$M_i = \sup_{[x_{i-1}, x_i]} f, \qquad m_i = \inf_{[x_{i-1}, x_i]} f,$$

the supremum and infimum of f over the i-th subinterval. The *upper* and *lower Riemann sums* of f over P are

$$U(P, f) = \sum_{i=1}^n M_i\,\Delta x_i, \qquad L(P, f) = \sum_{i=1}^n m_i\,\Delta x_i$$

(R 6.1). Since $m_i \le M_i$ on each subinterval, $L(P, f) \le U(P, f)$ for every P.

[1]Rudin builds the more general *Riemann–Stieltjes* integral $\int f\,d\alpha$ against an integrator α (R 6.1–R 6.2). Throughout this lecture we take $\alpha(x) = x$, which collapses $\int f\,d\alpha$ to the ordinary $\int f(x)\,dx$; every weight $\Delta\alpha_i$ below is just $\Delta x_i = x_i - x_{i-1}$. Boundedness is what makes the suprema and infima of 9.1.2 real numbers; both the closedness and the boundedness of $[a, b]$ matter.

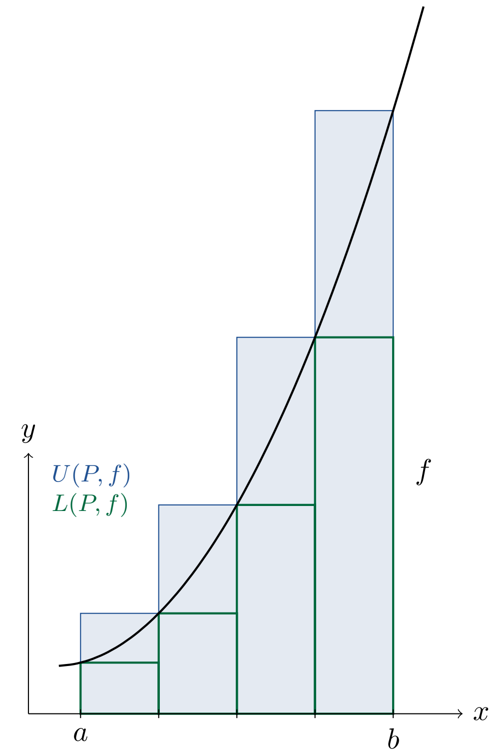

Figure 9.1.3. UPPER AND LOWER SUMS. The upper sum $U(P, f)$ is the area of the staircase whose steps reach the suprema M_i (tops above the curve); the lower sum $L(P, f)$ is the area of the staircase whose steps reach the infima m_i (tops below). The graph of f is caught between them, so $L(P, f) \le \int_a^b f \le U(P, f)$ whenever the integral exists.

Definition 9.1.4

Upper and lower integrals; Riemann integrable

The *upper* and *lower integrals* of f over $[a, b]$ are

$$\overline{\int_a^b} f = \inf_P U(P, f), \qquad \underline{\int_a^b} f = \sup_P L(P, f),$$

the infimum and supremum taken over all partitions P of $[a, b]$. We say f is *Riemann integrable*, written $f \in \mathcal{R}$, when these agree,

$$\underline{\int_a^b} f = \overline{\int_a^b} f,$$

and then their common value is the *integral* $\displaystyle\int_a^b f$ (R 6.2). Here $\mathcal{R} = \mathcal{R}[a, b]$ denotes the class of Riemann-integrable functions on $[a, b]$. If instead $\underline{\int} f \ne \overline{\int} f$, then $f \notin \mathcal{R}$.

Counterexample 9.1.5

The Dirichlet function

Let $f : [a, b] \to \mathbb{R}$ be the indicator of the irrationals,

$$f(x) = \begin{cases} 1, & x \notin \mathbb{Q}, \\ 0, & x \in \mathbb{Q}, \end{cases}$$

which is discontinuous at every point. Every subinterval $[x_{i-1}, x_i]$ contains both rationals and irrationals, so for any partition P,

$$M_i = 1, \qquad m_i = 0,$$

whence

$$U(P, f) = \sum_i M_i \, \Delta x_i = b - a = \overline{\int_a^b} f, \qquad L(P, f) = 0 = \underline{\int_a^b} f.$$

Since $b - a \neq 0$, the upper and lower integrals differ, so $f \notin \mathcal{R}$.[2]

FAILS: the claim that every bounded function is Riemann integrable.

9.2 Refinement and the Integrability Criterion

Definition 9.2.1

Refinement and common refinement

A partition P^* *refines* P if $P^* \supseteq P$ as sets of points; P^* then has all of P's division points and possibly more. Partitions are only *partially* ordered: for two partitions P_1, P_2 it may happen that $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. Their *common refinement* is the union $P_1 \cup P_2$, which refines both:

$$P_1 \cup P_2 \supseteq P_1, \qquad P_1 \cup P_2 \supseteq P_2$$

(R 6.3). The *mesh* of P is $|P| = \max_i \Delta x_i$, the width of its widest subinterval.[3]

Proposition 9.2.2

Refinement inequalities

If P^* refines P, then

$$L(P, f) \leq L(P^*, f) \qquad \text{and} \qquad U(P^*, f) \leq U(P, f)$$

(R 6.4). Refining a partition can only raise the lower sum and lower the upper sum.

Idea. Refining pushes the two staircases toward the curve, so the sums move together. It is enough to add a single point and iterate: passing to the smaller subintervals can only raise an

[2]The orientation is immaterial: putting $f = 1$ on $\mathbb{Q}$ instead swaps the names but leaves every $M_i = 1$, every $m_i = 0$, and the same gap $U - L = b - a$. The function reappears in 9.3.8, where the Lebesgue integral—grouping by value rather than by location—recovers a value Riemann cannot.

[3]Morally one wants $\int_a^b f = \lim_{|P| \to 0} R_n(f)$, the limit of the Riemann sums as the mesh shrinks. Refinement makes this precise without mentioning a limit: rather than driving $|P| \to 0$, we compare a partition with its refinements and show the upper and lower sums close in on each other (9.2.2).

infimum and lower a supremum.

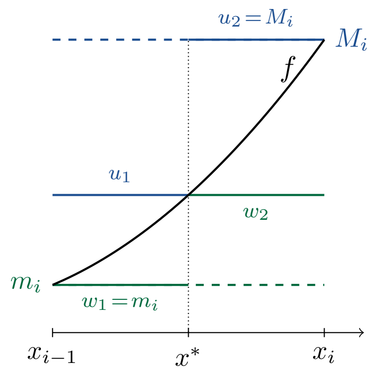

Figure 9.2.3. Adding one point to a partition. Inserting x^* splits $[x_{i-1}, x_i]$ into two pieces. The infimum over the whole, m_i, is at most the infima w_1, w_2 over the halves (inf over a smaller set rises), so the lower sum grows; the supremum over the whole, M_i, is at least the suprema u_1, u_2 (sup over a smaller set falls), so the upper sum shrinks.

Proof 9.2.2

Proof by Induction on the Number of Added Points.

(1) It suffices to treat $P^* = P \cup \{x^*\}$, one extra point: a general refinement is reached by adding the finitely many new points one at a time, and chaining the resulting inequalities gives the full statement.

(2) Say x^* falls in the i-th subinterval, $x_{i-1} \le x^* \le x_i$, splitting it into $[x_{i-1}, x^*]$ and $[x^*, x_i]$. Write

$$w_1 = \inf_{[x_{i-1},x^*]} f, \quad w_2 = \inf_{[x^*,x_i]} f, \qquad u_1 = \sup_{[x_{i-1},x^*]} f, \quad u_2 = \sup_{[x^*,x_i]} f.$$

(3) The infimum over a set is at most the infimum over any subset, and dually for suprema, so

$$m_i \le w_1, \qquad m_i \le w_2, \qquad M_i \ge u_1, \qquad M_i \ge u_2.$$

(4) All other subintervals are untouched, so on passing from P to P^* only the i-th term changes. Using $\Delta x_i = (x^* - x_{i-1}) + (x_i - x^*)$,

$$L(P^*, f) - L(P, f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i\,\Delta x_i \ge 0,$$

by Step (3), and likewise

$$U(P, f) - U(P^*, f) = M_i\,\Delta x_i - u_1(x^* - x_{i-1}) - u_2(x_i - x^*) \ge 0.$$

Hence $L(P, f) \leq L(P^*, f)$ and $U(P^*, f) \leq U(P, f)$.

■

[A10] cf. R 6.4; reduce to a single inserted point.

Corollary 9.2.4

Every lower sum is below every upper sum

For any two partitions P_1, P_2 of $[a, b]$,

$$L(P_1, f) \leq U(P_2, f)$$

(R 6.5).

Proof 9.2.4

Proof by Passing Through the Common Refinement.

(1) Let $P^* = P_1 \cup P_2$ be the common refinement, which refines both P_1 and P_2 (9.2.1). Then

$$L(P_1, f) \;\leq\; L(P_1 \cup P_2, f) \;\leq\; U(P_1 \cup P_2, f) \;\leq\; U(P_2, f).$$

(2) The first inequality is 9.2.2 applied to P_1 (lower sums rise under refinement), the middle is $L \leq U$ for the single partition P^* (9.1.2), and the last is 9.2.2 applied to P_2 (upper sums fall under refinement). Reading the ends gives the claim.

■

[Direct] cf. R 6.5.

Corollary 9.2.5

Lower integral does not exceed upper integral

For every bounded f on $[a, b]$,

$$\underline{\int_a^b} f \;\leq\; \overline{\int_a^b} f$$

(R 6.5).

Proof 9.2.5

Proof by a Supremum-then-Infimum Argument.

(1) Fix P_2. By 9.2.4, $L(P_1, f) \leq U(P_2, f)$ for every P_1, so $U(P_2, f)$ is an upper bound for the lower sums; taking the supremum over P_1,

$$\underline{\int_a^b} f = \sup_{P_1} L(P_1, f) \;\leq\; U(P_2, f).$$

(2) This now holds for every P_2, so $\underline{\int} f$ is a lower bound for the upper sums; taking the infimum over P_2,

$$\underline{\int_a^b} f \;\leq\; \inf_{P_2} U(P_2, f) = \overline{\int_a^b} f.$$

[A5] cf. R 6.5; take sup then inf in 9.2.4.

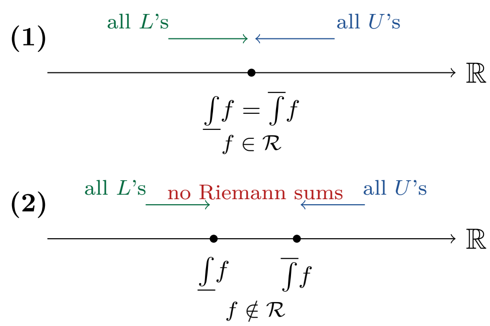

Figure 9.2.6. The two pictures. All lower sums lie to the left, all upper sums to the right, and $\underline{\int} f \leq \overline{\int} f$ separates them. **(1)** The two families meet at a single point: $\underline{\int} f = \overline{\int} f$ and $f \in \mathcal{R}$. **(2)** A gap remains, spanned by no Riemann sum: $\underline{\int} f < \overline{\int} f$ and $f \notin \mathcal{R}$.

Theorem 9.2.7

The integrability criterion

A bounded function f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P with

$$U(P, f) - L(P, f) < \varepsilon$$

(R 6.6).

Idea. Closeness of the two sums means closeness of the two integrals. If U and L can be brought within ε, the upper and lower integrals—trapped between them—are within ε too, for every ε, hence equal.

Proof 9.2.7

Proof of the Integrability Criterion.

(1) ($\Rightarrow$) Suppose $f \in \mathcal{R}$, so $\underline{\int} f = \overline{\int} f = \int_a^b f$. Let $\varepsilon > 0$. By the definitions of the infimum and supremum (9.1.4) there are partitions P_1, P_2 with

$$U(P_1, f) < \int_a^b f + \tfrac{\varepsilon}{2}, \qquad L(P_2, f) > \int_a^b f - \tfrac{\varepsilon}{2}.$$

Put $P = P_1 \cup P_2$. By 9.2.2, $U(P, f) \leq U(P_1, f)$ and $L(P, f) \geq L(P_2, f)$, so

$$U(P, f) - L(P, f) \ \leq\ U(P_1, f) - L(P_2, f) \ <\ \varepsilon.$$

(2) (⇐) Suppose that for every $\varepsilon > 0$ some P has $U(P,f) - L(P,f) < \varepsilon$. Since always

$$L(P,f) \;\le\; \underline{\int_a^b} f \;\le\; \overline{\int_a^b} f \;\le\; U(P,f),$$

the gap satisfies $0 \le \overline{\int} f - \underline{\int} f \le U(P,f) - L(P,f) < \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\overline{\int} f - \underline{\int} f = 0$, i.e. $f \in \mathcal{R}$.

■

[A1] cf. R 6.6; (⇐) is the two-pictures argument of 9.2.6.

Proposition 9.2.8

A working tool

Suppose $f \in \mathcal{R}$ on $[a,b]$, let $\varepsilon > 0$, and let P be a partition with $U(P,f) - L(P,f) < \varepsilon$. Then:

(a) the same inequality holds for every refinement $P^* \supseteq P$;

(b) for any sample points $s_i, t_i \in [x_{i-1}, x_i]$, $\displaystyle\sum_{i=1}^{n} |f(s_i) - f(t_i)|\,\Delta x_i < \varepsilon$;

(c) for any sample points $t_i \in [x_{i-1}, x_i]$, $\displaystyle\left|\sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f\right| < \varepsilon$.

This is the technical tool drawn on in the proofs below (R 6.7).

Proof 9.2.8

Proof by Sandwiching Between the Infima and Suprema.

(1) For (a), refining only shrinks U and grows L (9.2.2), so $U(P^*,f) - L(P^*,f) \le U(P,f) - L(P,f) < \varepsilon$.

(2) For (b), on each subinterval $f(s_i), f(t_i) \in [m_i, M_i]$, hence $|f(s_i) - f(t_i)| \le M_i - m_i$. Summing,

$$\sum_i |f(s_i) - f(t_i)|\,\Delta x_i \;\le\; \sum_i (M_i - m_i)\,\Delta x_i \;=\; U(P,f) - L(P,f) \;<\; \varepsilon.$$

(3) For (c), since $t_i \in [x_{i-1}, x_i]$ we have $m_i \le f(t_i) \le M_i$, so both $\sum_i f(t_i)\,\Delta x_i$ and $\int_a^b f$ lie in the interval $[L(P,f), U(P,f)]$, of length $< \varepsilon$. Two numbers in an interval of length $< \varepsilon$ differ by $< \varepsilon$.

■

[Direct] cf. R 6.7.

9.3 Classes of Integrable Functions

Theorem 9.3.1

Continuous functions are integrable

If $f : [a,b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}$ (R 6.8).

Idea. The key step is *uniform* continuity: on the compact interval $[a, b]$ a single δ controls the oscillation everywhere at once, so a fine enough partition makes every $M_i - m_i$ uniformly small.

Proof 9.3.1

Proof by Uniform Continuity on a Compact Interval.

(1) Since $[a, b]$ is compact, f is *uniformly continuous* (7.7.2). Given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

(2) Choose a partition P with $\Delta x_i < \delta$ for all i—for instance n equal subintervals with $(b - a)/n < \delta$. On each subinterval f attains its supremum and infimum (a continuous function on a compact set attains its extrema), so $M_i = f(\xi_i)$ and $m_i = f(\eta_i)$ for some $\xi_i, \eta_i \in [x_{i-1}, x_i]$ with $|\xi_i - \eta_i| < \delta$; hence

$$M_i - m_i < \frac{\varepsilon}{b - a} \qquad \text{for all } i.$$

(3) Therefore

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\, \Delta x_i < \frac{\varepsilon}{b - a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b - a}\,(b - a) = \varepsilon,$$

using $\sum_i \Delta x_i = b - a$. By the integrability criterion (9.2.7), $f \in \mathcal{R}$.

■

[A6+A1] cf. R 6.8; uniform continuity is 7.7.2.

Theorem 9.3.2

Monotone functions are integrable

If f is monotone on $[a, b]$, then $f \in \mathcal{R}$ (R 6.9). Such an f may have at most countably many jump discontinuities.

Idea. For a monotone f the differences $M_i - m_i$ telescope: on equal subintervals the total oscillation is just $f(b) - f(a)$, spread over n pieces, so it vanishes as $n \to \infty$.

Proof 9.3.2

Proof by Telescoping on an Equal Partition.

(1) Treat f monotone increasing. Let $\varepsilon > 0$ and take the equal partition with

$$\Delta x_i = \frac{b - a}{n}, \qquad n > \frac{(b - a)(f(b) - f(a))}{\varepsilon}.$$

(2) On $[x_{i-1}, x_i]$ monotonicity gives $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Pulling the con-

stant $\Delta x_i = (b-a)/n$ out of the sum, the differences telescope:

$$U(P,f) - L(P,f) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\Delta x_i = \frac{b-a}{n}\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))$$
$$= \frac{b-a}{n}(f(b) - f(a)).$$

(3) By the choice of n this is $< \varepsilon$. The criterion (9.2.7) gives $f \in \mathcal{R}$.

■

[Direct] cf. R 6.9; the decreasing case follows by applying the increasing case to $-f$.

Theorem 9.3.3

Finitely many discontinuities still permit integrability

If $f : [a,b] \to \mathbb{R}$ is bounded and has only *finitely many* discontinuities, then $f \in \mathcal{R}$ (R 6.10).

Idea. Finitely many points can be covered by intervals of arbitrarily small total length. Box the bad points into tiny intervals where the bounded f can misbehave by at most $2M$ each; on the compact remainder f is uniformly continuous, so the earlier argument applies. The two contributions to $U - L$ are then each $O(\varepsilon)$.

Proof 9.3.3

Constructive Proof by Covering the Discontinuities.

(1) Let $\varepsilon > 0$ and set $M := \sup_{[a,b]} |f| < \infty$, so $-M \le f \le M$. Let $E \subseteq [a,b]$ be the (finite) set of discontinuities. Cover E by finitely many disjoint closed intervals $[u_j, v_j]$, $j = 1, \ldots, m$, such that

(1) each point of $E \cap (a,b)$ is interior to some $[u_j, v_j]$; (2) $\sum_j (v_j - u_j) < \varepsilon$.

(2) The set

$$K := [a,b] \setminus \bigcup_j (u_j, v_j)$$

is closed in $[a,b]$ and hence compact, and by construction f is continuous on K. Therefore f is uniformly continuous on K (7.7.2): there is $\delta > 0$ with $x, y \in K$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

(3) Form a partition P such that: (i) every $u_j, v_j \in P$; (ii) no point of (u_j, v_j) lies in P; (iii) every subinterval not of the form $[u_j, v_j]$ has length $< \delta$. Split the upper-minus-lower sum according to the two flavours of subinterval:

$$U(P,f) - L(P,f) = \underbrace{\sum_{\text{covered}} (M_i - m_i)\,\Delta x_i}_{\text{the } [u_j,v_j]} + \underbrace{\sum_{\text{other}} (M_i - m_i)\,\Delta x_i}_{\text{length } <\delta}.$$

(4) On a covered subinterval $M_i - m_i \le 2M$, and these subintervals have total length $< \varepsilon$, so the first sum is $\le 2M\varepsilon$. On the others, the subinterval lies in K, where f is uniformly continuous, so $M_i - m_i \le \varepsilon$ by Step (2); the second sum is then $\le \varepsilon \sum_i \Delta x_i \le (b-a)\varepsilon$. Hence

$$U(P,f) - L(P,f) \;\le\; 2M\varepsilon + (b-a)\varepsilon.$$

The right side is an arbitrary multiple of ε, so $U - L$ can be made smaller than any prescribed positive number; by the criterion (9.2.7), $f \in \mathcal{R}$.

∎

[Constr] cf. R 6.10; a two-flavour partition splits $U - L$ into a covered part and a uniformly continuous part.

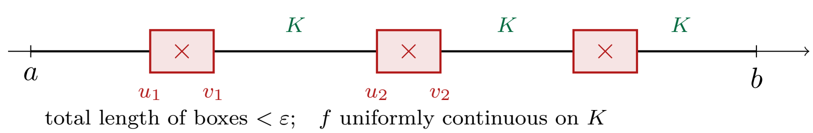

Figure 9.3.4. Covering the discontinuities. The discontinuities (marked $\times$) are boxed inside small closed intervals $[u_j, v_j]$ of total length $< \varepsilon$. On the compact remainder $K = [a,b] \setminus \bigcup_j (u_j, v_j)$ the function is continuous, hence uniformly continuous, so a partition fine away from the boxes controls $U - L$.

Theorem 9.3.5

The Lebesgue criterion

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}$ if and only if the set $E \subseteq [a,b]$ of discontinuities of f can be covered by countably many intervals of total length $< \varepsilon$ for every $\varepsilon > 0$—that is, E has *length (measure) zero*. This is the full integrability criterion; the lecture proved only the finite case (9.3.3).[4]

Proof 9.3.5

The General Lebesgue Criterion.

↪ Stated as the complete characterization of $\mathcal{R}$; its proof, and the precise notion of measure zero, lie beyond Ch. 6 (Rudin Ch. 11). The lecture established only the finite-discontinuity case (9.3.3).

Example 9.3.6

The Cantor set: uncountable, yet length zero

[4] "Length zero" is the intuitive face of *Lebesgue measure zero*: a set E is null if for every $\varepsilon > 0$ it fits inside a countable union of intervals whose lengths total less than ε. The finite cover of 9.3.3 is the simplest instance. The notion—and this characterization of $\mathcal{R}$—is made precise in the measure theory of Ch. 11 (R 11.33).

Build the middle-thirds set by deleting open middle thirds: $C_0 = [0,1]$; C_1 removes the middle third, leaving two intervals; C_2 removes the middle third of each, leaving four; and so on. The *Cantor set* is the intersection

$$C = \bigcap_{n\geq 0} C_n.$$

It is *uncountable*: a point of C is determined by an infinite sequence of left/right choices, one at each stage, so by the binary-choice argument of 2.1.10 there are uncountably many. Yet C has length zero,[5] since the deleted thirds exhaust the unit length,

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\cdots = \frac{1}{3}\sum_{k=0}^{\infty}\Big(\frac{2}{3}\Big)^k = 1, \qquad \text{so} \qquad 1-\frac{1}{3}-\frac{2}{9}-\frac{4}{27}-\cdots = 0.$$

By the Lebesgue criterion (9.3.5), a bounded function discontinuous *exactly* on C is still Riemann integrable—the sharp contrast with the Dirichlet function (9.1.5), which is discontinuous on all of $[a,b]$, a set of full length $b-a$ and far from null.

Figure 9.3.7. THE CANTOR MIDDLE-THIRDS CONSTRUCTION. Each stage deletes the open middle third of every surviving interval: C_0 is one interval, C_1 two, C_2 four. The total deleted length is $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots = 1$, so $C = \bigcap_n C_n$ has length zero—while remaining uncountable.

Remark 9.3.8

Toward the Lebesgue integral

Where Riemann sums by *location*, Lebesgue sums by *value*. For the Dirichlet function this recovers an integral Riemann cannot assign:

$$\int_{[a,b]} f = \int_{[a,b]\cap\mathbb{Q}} f + \int_{[a,b]\setminus\mathbb{Q}} f = 0\cdot\ell\big([a,b]\cap\mathbb{Q}\big) + 1\cdot\ell\big([a,b]\setminus\mathbb{Q}\big) = b-a,$$

since the rationals have length 0 and the irrationals length $b-a$. Riemann cannot do this, because $\{\mathbb{Q},\ \mathbb{R}\setminus\mathbb{Q}\}$ is "not a partition" of $[a,b]$ in his sense—it splits by value, not by intervals. This is the move the epigraph describes, and it belongs to Ch. 11.

9.4 Properties and the Fundamental Theorem

The remaining results were stated in lecture with their proofs *deferred to Lecture 10*; we record them here as stated facts so the lecture is complete and the Fundamental Theorem is in place. Each proof is supplied next time.

[5] The geometric-series computation is the intuitive face of a rigorous fact: C has *Lebesgue measure zero*, by the countable additivity of measure (Ch. 11). That machinery is not needed here—the elementary length bound suffices—but it is what makes "length zero" precise.

Theorem 9.4.1

Composition with a continuous function

If $f \in \mathcal{R}$ on $[a, b]$ with $m \leq f \leq M$, and φ is continuous on the closed interval $[m, M]$, then $\varphi \circ f \in \mathcal{R}$ (R 6.11).

Proof 9.4.1

Composition Preserves Integrability.

↪ Proof deferred to Lecture 10.

Proposition 9.4.2

Properties of the integral

For $f, g \in \mathcal{R}$ on $[a, b]$:

(a) *Linearity:* $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ and $\int_a^b cf = c\int_a^b f$ for $c \in \mathbb{R}$.

(b) *Monotonicity:* if $f \leq g$ on $[a, b]$ then $\int_a^b f \leq \int_a^b g$; and if $|f(x)| \leq M$ then $\left|\int_a^b f\right| \leq M(b-a)$.

(c) *Products:* $fg \in \mathcal{R}$.

(d) *Absolute value:* $|f| \in \mathcal{R}$, and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

(e) *Additivity:* for $c \in [a, b]$, $\int_a^b f = \int_a^c f + \int_c^b f$.

These are Rudin's basic properties of the integral (R 6.12–R 6.13).

Proof 9.4.2

Properties of the Integral.

↪ Proofs deferred to Lecture 10.

Theorem 9.4.3

Fundamental Theorem of Calculus, I

Let $f \in \mathcal{R}$ on $[a, b]$. Then the "antiderivative"

$$F(x) = \int_a^x f(t)\,dt$$

is uniformly continuous on $[a, b]$; and if f is continuous at a point $t_0 \in [a, b]$, then F is differentiable at t_0 with

$$F'(t_0) = f(t_0)$$

(R 6.20).

Proof 9.4.3

The Fundamental Theorem, I.

↪ Proof deferred to Lecture 10.

Theorem 9.4.4

Fundamental Theorem of Calculus, II

Let $f \in \mathcal{R}$ on $[a, b]$, and suppose there is $F : [a, b] \to \mathbb{R}$ with $F' = f$ on $[a, b]$. Then

$$\int_a^b f = F(b) - F(a)$$

(R 6.21).

Proof 9.4.4

The Fundamental Theorem, II.

$\hookrightarrow$ Proof deferred to Lecture 10.

Remark 9.4.5

Integration smooths, differentiation roughens

A closing thought from lecture. Integration makes functions *better*: it carries a merely integrable f to a continuous F, and a continuous f to a differentiable one (9.4.3). Differentiation runs the other way, making functions *worse*—a differentiable function can have a discontinuous derivative. The two operations undo each other, which is exactly the content of the Fundamental Theorem (9.4.3, 9.4.4).

Index of Objects