LECTURE 7

Function Limits, Continuity, and the Topology of Continuous Maps

MATH S4061 | Introduction to Modern Analysis I

Thursday, 18 June 2026

Natura non facit saltus.

Gottfried Wilhelm Leibniz, New Essays IV.16

Contents of this lecture

Overview. This lecture begins Chapter 4 by replacing the sequential limit $p_n \to p$ with the function limit $\lim_{x\to p} f(x) = q$ in metric spaces. The ε–δ definition is tied back to sequences, then specialized to continuity at a point and continuity on a set. With the open-set characterization in hand, continuity becomes a topological operation: preimages of open sets are open, compositions are continuous, compact sets have compact images, connected sets have connected images, and the classical extreme-value and intermediate-value theorems become consequences. The lecture closes with homeomorphisms and the compactness theorem that upgrades pointwise continuity to uniform continuity. The companion text is Rudin, Chapter 4, especially R 4.1–R 4.26.

7.1 Function Limits

Theorem 7.1.1

Extreme values on a closed interval

If $f : [a,b] \to \mathbb{R}$ is continuous, then f attains a maximum and a minimum: there exist $x, y \in [a,b]$ such that

$$f(x) \le f(z) \le f(y) \qquad \text{for every } z \in [a,b].$$

This is the interval case of Rudin's extreme-value theorem (R 4.16); the proof is deferred until compactness of images is available in 7.5.2.

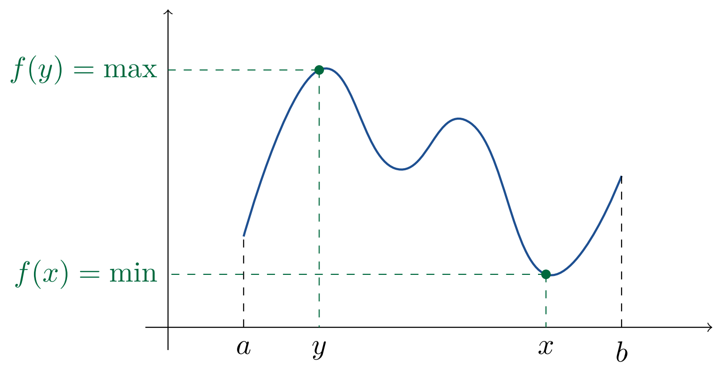


Figure 7.1.2. THE CALCULUS PICTURE OF EXTREMA. A continuous real-valued function on a closed interval cannot miss its highest and lowest values. The rigorous proof will come from compactness of the image (7.5.2).

Remark 7.1.3

Sequences as functions

For a sequence $\{p_n\}$ in a metric space X, writing $p_n \to p$ means $\lim_{n\to\infty} p_n = p$ in the ε–N sense. A sequence is simply a function

$$f : \mathbb{N} \to X, \qquad n \mapsto p_n,$$

and asks what happens as n gets large (cf. R 2.7, R 3.1).

Definition 7.1.4

Function limit in metric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$, let p be a cluster point of E, and let $f : E \to Y$. We say that the *limit of f as $x \to p$ is q* (R 4.1) if for every $\varepsilon > 0$ there exists $\delta_{p,\varepsilon} > 0$ such that

$$0 < d_X(x,p) < \delta_{p,\varepsilon} \quad \Longrightarrow \quad d_Y\big(f(x), q\big) < \varepsilon.$$

We write

$$f(x) \to q \quad \text{as } x \to p, \qquad \text{or} \qquad \lim_{x\to p} f(x) = q.$$

The condition $0 < d_X(x,p)$ forces $x \neq p$; therefore $f(p)$ need not be defined, and even when it is defined its value does not affect this limit.[1]

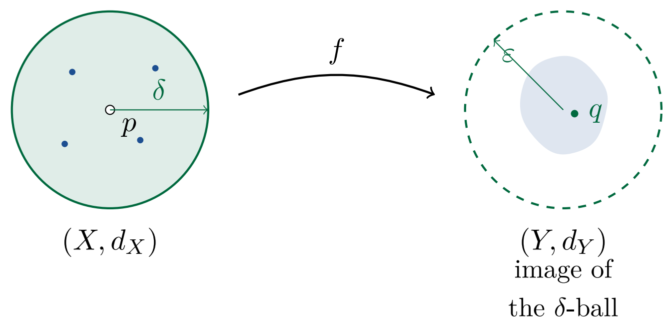


Figure 7.1.5. THE ε–δ PICTURE. The punctured δ-ball around p in the domain is carried inside the ε-ball around q in the target.

Proposition 7.1.6

Sequential criterion for function limits

Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$, let $f : E \to Y$, and let p be a cluster point of E. Then

$$\lim_{x \to p} f(x) = q \quad \iff \quad f(x_n) \to q \text{ for every sequence } \{x_n\} \subseteq E \setminus \{p\} \text{ with } x_n \to p.$$

This is Rudin's sequential characterization of function limits (R 4.2).

Proof 7.1.6

Proof by the ε–δ Definition and Its Contrapositive.

(1) Suppose $\lim_{x \to p} f(x) = q$ and let $\{x_n\} \subseteq E \setminus \{p\}$ satisfy $x_n \to p$. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$0 < d_X(x,p) < \delta \quad \Longrightarrow \quad d_Y(f(x), q) < \varepsilon.$$

Since $x_n \to p$, for all large n one has $d_X(x_n, p) < \delta$, and the punctured condition is automatic because $x_n \neq p$. Hence $d_Y(f(x_n), q) < \varepsilon$ eventually, so $f(x_n) \to q$.

(2) Conversely, prove the contrapositive. If the function limit fails, then for some $\varepsilon_0 > 0$ every $\delta > 0$ fails. Taking $\delta = 1/n$, choose $x_n \in E$ with

$$0 < d_X(x_n, p) < \frac{1}{n} \qquad \text{but} \qquad d_Y(f(x_n), q) \geq \varepsilon_0.$$

[1] A limit is a statement about the *approach* to p, never about the value at p. The punctured condition $0 < d_X(x,p)$ deliberately looks away from $x = p$, so $f(p)$ may be undefined, or defined "wrongly," without changing $\lim_{x \to p} f$. The gap between the value the approach predicts and the value actually assigned at p is exactly what a removable discontinuity (7.3.7) records.

Then $x_n \to p$, while $f(x_n)$ does not converge to q. This contradicts the sequential condition, so the function limit holds.

■

[A2+A11] cf. R 4.2; use $\delta_n = 1/n$ in the reverse direction.

The logic is: if A is the function-limit statement and B is the sequential statement, prove $A \Rightarrow B$ and $\neg A \Rightarrow \neg B$.

Proposition 7.1.7

Arithmetic of function limits

Let $E \subseteq X$, let p be a cluster point of E, and let $f, g : E \to \mathbb{R}$ satisfy $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then

$$\lim_{x\to p}(f+g)(x) = A + B, \qquad \lim_{x\to p}(fg)(x) = AB, \qquad \lim_{x\to p}\frac{f}{g}(x) = \frac{A}{B} \quad (B \neq 0).$$

The same statements hold coordinatewise for $\mathbb{R}^k$-valued f, g, and for their inner product $f \cdot g$. This is the function-limit form of the algebraic limit laws (R 4.4); because limits of functions reduce to limits of sequences, the Chapter 3 laws carry over verbatim.

Proof 7.1.7

Proof by the Sequential Criterion and the Chapter 3 Limit Laws.

(1) Let $\{x_n\} \subseteq E \setminus \{p\}$ be any sequence with $x_n \to p$. By the sequential criterion 7.1.6, the hypotheses give

$$f(x_n) \to A, \qquad g(x_n) \to B.$$

(2) The algebraic limit laws for sequences (R 3.3) then apply termwise:

$$f(x_n) + g(x_n) \to A + B, \qquad f(x_n)\,g(x_n) \to AB, \qquad \frac{f(x_n)}{g(x_n)} \to \frac{A}{B} \quad (B \neq 0),$$

where in the quotient case $B \neq 0$ keeps the terms eventually defined and bounded away from 0.

(3) Since these hold for *every* such sequence $x_n \to p$, the reverse direction of 7.1.6 converts each one back into the corresponding function limit. This proves the three identities.

(4) For $\mathbb{R}^k$-valued f, g the same argument runs in each coordinate, so the vector limits hold coordinatewise; and since $f \cdot g = \sum_{i=1}^{k} f_i\, g_i$ is built from the real products and sums just treated, $\lim_{x\to p}(f \cdot g)(x) = A \cdot B$ as well.

■

[A2] cf. R 4.4; uses 7.1.6 and the sequence limit laws R 3.3.

7.2 Continuity and Composition

Definition 7.2.1

Continuity at a point

Let $f : E \to Y$ and let $p \in E$. The function f is *continuous at* p (R 4.5) if

$$\lim_{x \to p} f(x) = f(p).$$

Equivalently, for every $\varepsilon > 0$ there exists $\delta_{\varepsilon,p} > 0$ such that

$$d_X(x,p) < \delta_{\varepsilon,p} \quad \Longrightarrow \quad d_Y\big(f(x), f(p)\big) < \varepsilon.$$

For continuity the ball around p is not punctured: the point $x = p$ is allowed, and the limit value is required to be exactly $f(p)$ (cf. R 4.6 when p is a cluster point).

Definition 7.2.2

Continuity on a set

A function $f : E \to Y$ is *continuous on* E if it is continuous at every $p \in E$ (R 4.5).

Remark 7.2.3

Isolated points

If p is an isolated point of E, then f is continuous at p vacuously: for a sufficiently small ball, there are no points of E near p except p itself (cf. R 4.5).

Proposition 7.2.4

Composition of continuous functions

If $f : X \to Y$ is continuous at x and $g : Y \to Z$ is continuous at $f(x)$, then $g \circ f : X \to Z$ is continuous at x. Consequently, the composition of continuous functions is continuous (R 4.7).

Proof 7.2.4

Proof by an ε–η–δ Chase.

(1) Let $\varepsilon > 0$. Since g is continuous at $f(x)$, there exists $\eta > 0$ such that

$$d_Y(y, f(x)) < \eta \quad \Longrightarrow \quad d_Z\big(g(y), g(f(x))\big) < \varepsilon.$$

(2) Since f is continuous at x, there exists $\delta > 0$ such that

$$d_X(u,x) < \delta \quad \Longrightarrow \quad d_Y\big(f(u), f(x)\big) < \eta.$$

(3) Combining the two implications, $d_X(u,x) < \delta$ implies

$$d_Z\big((g \circ f)(u), (g \circ f)(x)\big) < \varepsilon,$$

so $g \circ f$ is continuous at x. ■

[A3] cf. R 4.7.

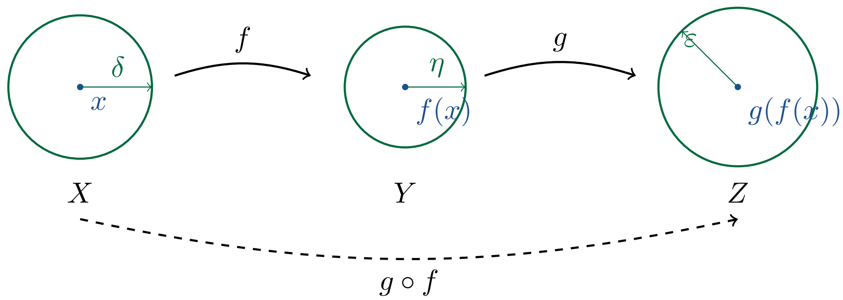


Figure 7.2.5. THE COMPOSITION CHASE. Continuity of g supplies the η-ball in Y; continuity of f supplies the δ-ball in X whose image lands inside it.

Remark 7.2.6

Sequential proof of composition

The sequential criterion gives the same result at a glance:

$$x_n \to x \quad \Longrightarrow \quad f(x_n) \to f(x) \quad \Longrightarrow \quad g(f(x_n)) \to g(f(x)).$$

This is the sequential version of R 4.7, using R 4.2.

7.3 Examples and Discontinuities

Example 7.3.1

Basic continuous real functions

The constant function $f(x) \equiv c$ is continuous: given $\varepsilon > 0$, every value of f already lies in $(c-\varepsilon, c+\varepsilon)$, so any $\delta > 0$ works. The identity $f(x) = x$ is continuous by taking $\delta = \varepsilon$. Building from constants and the identity through the algebra of limits and composition, every polynomial is continuous, and every rational function $p(x)/q(x)$ is continuous where $q(x) \neq 0$ (cf. R 4.9–R 4.11).

Proposition 7.3.2

The reciprocal is continuous on $(0, \infty)$

The function $f(t) = 1/t$ is continuous on $(0, \infty)$, but the required δ depends on both the point x and ε. This is a concrete ε–δ proof of the reciprocal case covered abstractly by R 4.9.

Proof 7.3.2

Direct ε–δ Proof for the Reciprocal.

(1) Fix $x > 0$ and $\varepsilon > 0$. If $|t - x| < \delta$, then

$$\left|\frac{1}{t} - \frac{1}{x}\right| = \frac{|t-x|}{tx}.$$

(2) First keep t away from 0. If $\delta \le x/2$, then $t > x/2$, so $tx > x^2/2$ and

$$\left|\frac{1}{t} - \frac{1}{x}\right| < \frac{\delta}{x^2/2} = \frac{2\delta}{x^2}.$$

(3) Therefore it is enough to require $\delta < \varepsilon x^2/2$. One may take

$$\delta(x, \varepsilon) = \min\left(\frac{x}{2}, \frac{\varepsilon x^2}{2}\right).$$

This proves continuity at x. The factor x^2 shows why δ shrinks as $x \to 0$. ∎

[A3] cf. R 4.9; the point-dependence is the point of the example.

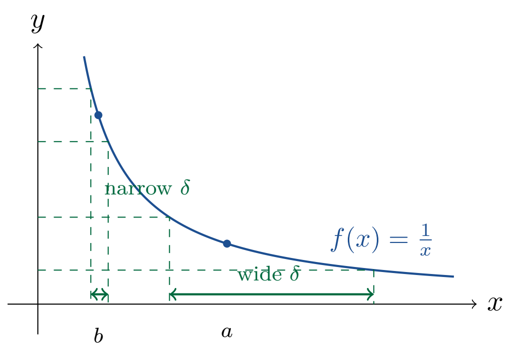


Figure 7.3.3. Point-dependent δ for $1/x$. The same ε-band in the target pulls back to a narrow input interval where the graph is steep and a wider interval where the graph is flat.

Remark 7.3.4

Target and domain roles

The ε always measures closeness in the target Y; the δ always measures closeness in the domain X. In 7.3.2 the roles do not swap. Only the size of δ changes with the base point (cf. R 4.5–R 4.8).

Remark 7.3.5

Power series and familiar functions

Power series define continuous functions inside their radius of convergence. In particular,

$$e^x = \sum_{n \ge 0} \frac{x^n}{n!}$$

is continuous, and similarly $\sin x$, $\cos x$, $\tan x$, and the other familiar functions are continuous on their natural domains. Compare the radius-of-convergence discussion in 6.6.11 and Rudin's power-series radius theorem R 3.39.

Example 7.3.6

Standard discontinuities

The floor function $\lfloor x \rfloor$ is discontinuous at each integer. The piecewise function

$$f(x) = \begin{cases} e^x, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

is discontinuous at 0, since $\lim_{x \to 0^+} f(x) = 1$ while $\lim_{x \to 0^-} f(x) = 0$. Compare Rudin's catalogue of discontinuity examples in R 4.27.

Definition 7.3.7

Type I and Type II discontinuities

For a real function discontinuous at x_0, the one-sided limits classify the discontinuity.

- Type I: both one-sided limits exist, but either they differ (a jump) or they agree without matching $f(x_0)$ (a removable discontinuity).
- Type II: at least one one-sided limit fails to exist. This is often called an essential discontinuity.

This matches Rudin's first-kind/second-kind terminology (R 4.25–R 4.26).

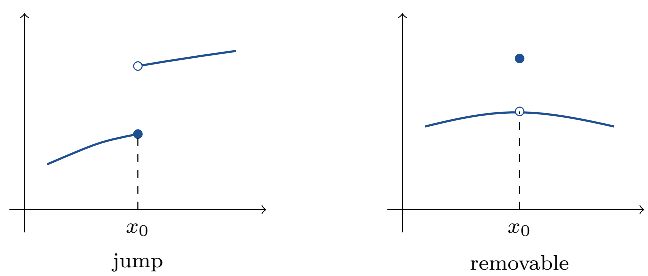


Figure 7.3.8. TYPE I DISCONTINUITIES. A jump has different one-sided limits; a removable discontinuity has a hole at the limiting value and the function value placed elsewhere.

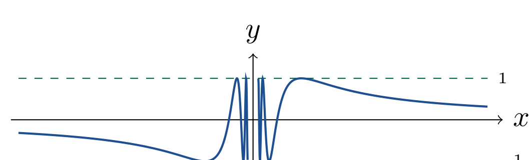
Example 7.3.9

The essential discontinuity of $\sin(1/x)$

The function

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

has no limit at 0. Near 0 the graph oscillates infinitely often between -1 and 1; for different sequences $x_n \to 0$, the values $f(x_n)$ can approach different limits (cf. R 4.27).



$f(x) = \sin(1/x)$ near 0

Figure 7.3.10. A Type II discontinuity. The graph of $\sin(1/x)$ oscillates indefinitely near 0, so no single limiting value can govern all approaches to 0.

Proposition 7.3.11

Projections and the norm are continuous

The coordinate projections and the norm on $\mathbb{R}^k$ are continuous; indeed each is Lipschitz, so $\delta = \varepsilon$ works at every point. Explicitly, for the i-th projection $\varphi_i : \mathbb{R}^k \to \mathbb{R}$, $\varphi_i(x_1, \ldots, x_k) = x_i$, and for the norm $\|\cdot\| : \mathbb{R}^k \to \mathbb{R}$,

$$|\varphi_i(x) - \varphi_i(y)| \le \|x - y\|, \qquad \big|\, \|x\| - \|y\| \,\big| \le \|x - y\|.$$

These are the building blocks behind the continuity of polynomials and rational functions on $\mathbb{R}^k$ (R 4.10).

Proof 7.3.11

Proof by Two Lipschitz Estimates.

(1) For the projection, the i-th coordinate difference is dominated by the full Euclidean distance:

$$|\varphi_i(x) - \varphi_i(y)| = |x_i - y_i| \le \Big(\sum_{j=1}^{k} |x_j - y_j|^2\Big)^{1/2} = \|x - y\|.$$

Hence $\|x - y\| < \varepsilon$ forces $|\varphi_i(x) - \varphi_i(y)| < \varepsilon$, so $\delta = \varepsilon$ proves continuity of φ_i at every point.

(2) For the norm, the reverse triangle inequality gives

$$\big|\, \|x\| - \|y\| \,\big| \le \|x - y\|,$$

so again $\|x - y\| < \varepsilon$ forces $\big|\, \|x\| - \|y\| \,\big| < \varepsilon$, and $\delta = \varepsilon$ proves continuity of $\|\cdot\|$.

■

[A3] cf. R 4.10; the second bound is the reverse triangle inequality.

Remark 7.3.12

Coordinatewise reduction of vector-valued continuity

A map $f : X \to \mathbb{R}^k$ is continuous if and only if each of its coordinate functions $f_i : X \to \mathbb{R}$ is continuous (R 4.10). One direction is immediate from 7.3.11: if f is continuous then $f_i = \varphi_i \circ f$ is a composition of continuous maps (7.2.4), hence continuous. Conversely, if every f_i is continuous then $\|f(x) - f(p)\|^2 = \sum_{i=1}^{k} |f_i(x) - f_i(p)|^2 \to 0$ as $x \to p$, so f is continuous. This reduces vector-valued continuity to the real-valued case.

7.4 Open-Set Characterizations

Proposition 7.4.1

Sequential characterization of continuity

A function $f : E \to Y$ is continuous at $p \in E$ if and only if for every sequence $\{x_n\} \subseteq E$ with $x_n \to p$, one has $f(x_n) \to f(p)$. This specializes Rudin's sequential limit criterion to continuity (cf. R 4.2, R 4.6).

Proof 7.4.1

Proof by Specializing the Limit Criterion.

(1) If f is continuous at p, apply 7.1.6 with $q = f(p)$; the non-punctured case causes no issue, since a sequence equal to p along a subsequence already has image equal to $f(p)$ along that subsequence.

(2) Conversely, if continuity fails at p, then for some $\varepsilon_0 > 0$ every $\delta > 0$ fails. With $\delta = 1/n$, choose $x_n \in E$ such that

$$d_X(x_n, p) < \frac{1}{n} \qquad \text{but} \qquad d_Y(f(x_n), f(p)) \geq \varepsilon_0.$$

Then $x_n \to p$, but $f(x_n)$ does not converge to $f(p)$.

■

[A2+A11] cf. R 4.2; the reverse direction again uses $\delta_n = 1/n$.

Proposition 7.4.2

Continuity at a point via neighbourhoods

The function $f : X \to Y$ is continuous at p if and only if for every open set $V \subseteq Y$ containing $f(p)$, there is an open set $U \subseteq X$ containing p such that

$$f(U) \subseteq V \qquad \text{equivalently} \qquad U \subseteq f^{-1}(V).$$

This is the pointwise neighbourhood form underlying Rudin's global open-set characterization (R 4.8).

Proof 7.4.2

Proof by Translating Balls into Open Neighbourhoods.

(1) If f is continuous at p and $V \subseteq Y$ is open with $f(p) \in V$, choose $\varepsilon > 0$ such that $B_\varepsilon(f(p)) \subseteq V$. Continuity gives $\delta > 0$ such that

$$f(B_\delta(p)) \subseteq B_\varepsilon(f(p)) \subseteq V.$$

Taking $U = B_\delta(p)$ gives the required neighbourhood.

(2) Conversely, take $V = B_\varepsilon(f(p))$. The neighbourhood condition gives an open $U \ni p$ with $f(U) \subseteq V$. Since U is open, some $B_\delta(p) \subseteq U$, and hence

$$f(B_\delta(p)) \subseteq B_\varepsilon(f(p)).$$

This is exactly the ε–δ condition for continuity at p.

■

[A3]

Proposition 7.4.3

Global open-set characterization of continuity

A function $f : X \to Y$ is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$.[2] This is Rudin's open-set characterization of continuity (R 4.8).

Proof 7.4.3

Proof by Pulling Back Open Balls.

(1) Suppose f is continuous and let $V \subseteq Y$ be open. To show $f^{-1}(V)$ is open, take $x \in f^{-1}(V)$. Since V is open, there is $\varepsilon > 0$ with $B_\varepsilon(f(x)) \subseteq V$. Continuity at x gives $\delta > 0$ with

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq V.$$

Hence $B_\delta(x) \subseteq f^{-1}(V)$, so every point of $f^{-1}(V)$ is interior.

(2) Conversely, fix $x \in X$ and $\varepsilon > 0$. The ball $B_\varepsilon(f(x))$ is open in Y, so $f^{-1}(B_\varepsilon(f(x)))$ is open in X by hypothesis. Since x lies in this preimage, there is $\delta > 0$ with

$$B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))).$$

Thus $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$, so f is continuous at x. Since x was arbitrary, f is continuous on X.

■

[Direct] cf. R 4.8.

Proposition 7.4.4

Composition via preimages

The composition of continuous functions is continuous. This is a second proof of R 4.7, using R 4.8.

Proof 7.4.4

Proof by the Open-Set Characterization.

(1) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f and g continuous, and let $V \subseteq Z$ be open.

(2) Since g is continuous, $g^{-1}(V)$ is open in Y. Since f is continuous, $f^{-1}(g^{-1}(V))$ is open in X.

[2] This is what continuity *really* is; the ε–δ definition is a local symptom of it. Continuity means the map never tears the space open—pull any open target set back and the domain set is still open. It has to be the *preimage*, not the image: preimages respect unions, intersections, and complements exactly, so they carry the open-set bookkeeping that images scramble. Read this way, continuity of compositions and the compact- and connected-image theorems drop out almost in one line (7.4.4, 7.5.1, 7.5.4).

(3) But preimage commutes with composition:

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)).$$

Thus the preimage under $g \circ f$ of every open set is open, so $g \circ f$ is continuous. ∎

[Direct] a second proof of 7.2.4; cf. R 4.8.

This is the clean topological proof of composition: it works because preimages, not images, commute exactly with composition.

Remark 7.4.5

Pugh's moral

The sequential criterion is often the easiest way to check continuity at a glance: to test continuity at p, push every sequence $x_n \to p$ through the function and see whether $f(x_n) \to f(p)$ (cf. R 4.2).

7.5 Continuous Images of Compact and Connected Sets

Theorem 7.5.1

Continuous image of a compact set

If $f : X \to Y$ is continuous and X is compact, then the image

$$f(X) = \{y \in Y : \exists\, x \in X \text{ with } f(x) = y\}$$

is compact in Y (R 4.14).[3]

Proof 7.5.1

Proof by Pulling Back an Open Cover.

(1) Let $\{V_\alpha\}$ be an open cover of $f(X)$, with each V_α open in Y. Since f is continuous, each $f^{-1}(V_\alpha)$ is open in X.

(2) These preimages cover X:

$$X = \bigcup_\alpha f^{-1}(V_\alpha).$$

Indeed, if $x \in X$, then $f(x) \in f(X)$, so $f(x) \in V_\alpha$ for some α, and hence $x \in f^{-1}(V_\alpha)$.

[3] Compactness and continuity are made for each other. Compactness speaks in open covers; continuity turns a cover of the image into a cover of the domain by pulling each open set back (7.4.3), where a finite subcover is guaranteed—and pushing forward gives a finite subcover of the image. The most useful corollary drops out immediately: a continuous real function on a compact set attains its maximum and minimum (7.5.2).

(3) By compactness of X, choose finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$X = f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}).$$

Applying f, the sets $V_{\alpha_1}, \ldots, V_{\alpha_n}$ cover $f(X)$. Thus every open cover of $f(X)$ has a finite subcover, so $f(X)$ is compact.

■

[A6] cf. R 4.14; uses 7.4.3.

Corollary 7.5.2

Extreme value theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, then it attains a maximum and a minimum: there exist $m, M \in [a, b]$ such that

$$f(m) \leq f(x) \leq f(M) \qquad \text{for all } x \in [a, b].$$

This is Rudin's extreme-value theorem for real-valued continuous functions on compact domains (R 4.16).

Proof 7.5.2

Proof by Compactness of the Image.

(1) The interval $[a, b]$ is compact, so $f([a, b])$ is compact by 7.5.1. By Heine–Borel in $\mathbb{R}$, it is closed and bounded.

(2) Since $f([a, b])$ is bounded, the real numbers

$$\alpha = \inf f([a, b]), \qquad \beta = \sup f([a, b])$$

exist.

(3) Since $f([a, b])$ is closed, it contains its infimum and supremum. Thus $\alpha, \beta \in f([a, b])$, so there are $m, M \in [a, b]$ with

$$f(m) = \alpha, \qquad f(M) = \beta.$$

Therefore $f(m) \leq f(x) \leq f(M)$ for every $x \in [a, b]$.

■

[A6+A5] cf. R 4.16; uses 7.5.1 and Heine–Borel.

Definition 7.5.3

Connected metric space

A metric space (X, d) is *connected* if it cannot be written as $X = A \cup B$ where

$$A \neq \emptyset, \qquad B \neq \emptyset, \qquad A \cap B = \emptyset, \qquad A, B \text{ are open}.$$

Such a decomposition is called a *separation*. Equivalently, the only clopen subsets of X are $\emptyset$ and X itself.[4] In $\mathbb{R}$, the connected subsets are exactly the intervals (cf. R 2.45, R 2.47).

Theorem 7.5.4

Continuous image of a connected set

If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected (R 4.22).

Proof 7.5.4

Proof by Pulling Back a Separation.

(1) Suppose for contradiction that $f(X) = A \cup B$ is a separation of $f(X)$: the sets A and B are nonempty, disjoint, and open in the subspace $f(X)$.

(2) Then $f^{-1}(A)$ and $f^{-1}(B)$ cover X, are disjoint, and are open in X by continuity. They are also nonempty: because $A, B \subseteq f(X)$ are nonempty, each contains some value $f(x)$.

(3) Thus $X = f^{-1}(A) \cup f^{-1}(B)$ is a separation of X, contradicting connectedness. Therefore $f(X)$ is connected.

■

[A9] cf. R 4.22; uses 7.4.3.

Corollary 7.5.5

Intermediate value theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and assume, without loss of generality, that $f(a) \leq f(b)$. Then for every $y \in [f(a), f(b)]$ there exists $x \in [a, b]$ with $f(x) = y$. This is the intermediate-value theorem (R 4.23).

Proof 7.5.5

Proof by Connectedness of the Image.

(1) The interval $[a, b]$ is connected, so $f([a, b])$ is connected by 7.5.4. Since connected subsets of $\mathbb{R}$ are intervals, $f([a, b])$ is an interval.

(2) This interval contains $f(a)$ and $f(b)$, and therefore contains every $y \in [f(a), f(b)]$. Hence for each such y there is $x \in [a, b]$ with $f(x) = y$.

■

[Direct] cf. R 4.23; compactness gives closed and bounded; connectedness gives an interval.

The structural mirror is useful: continuity preserves compactness, giving maxima and minima; continuity preserves connectedness, giving intermediate values.

[4] Read “connected” as “all one piece”: you cannot break X into two nonempty open parts with nothing bridging them. The clopen rephrasing is the test that gets used—a set that is both open and closed has cleanly detached from its complement, so any such set beyond $\emptyset$ and X *is* a break. On the line this is just the statement that the connected sets are the intervals: anything with a gap splits at the gap.

7.6 Homeomorphisms

Two metric spaces are considered topologically the same when there is a bijection between them that preserves the open-set structure in both directions.

Definition 7.6.1

Homeomorphism

A *homeomorphism* is a continuous bijection $f : X \to Y$ whose inverse $f^{-1} : Y \to X$ is also continuous. If such an f exists, then X and Y are *homeomorphic*. Rudin uses the compact-domain theorem for continuous bijections in R 4.17.

Theorem 7.6.2

A continuous bijection from a compact domain is a homeomorphism

Let $f : X \to Y$ be continuous and bijective. If X is compact, then $f^{-1} : Y \to X$ is continuous, so f is a homeomorphism (R 4.17).

Proof 7.6.2

Proof by Showing the Inverse Pulls Back Opens to Opens.

(1) To prove f^{-1} is continuous, it is enough to show that for every open $V \subseteq X$, the preimage of V under f^{-1} is open in Y. Since f is a bijection,

$$(f^{-1})^{-1}(V) = f(V).$$

(2) Let $V \subseteq X$ be open. Then V^c is closed in X, and since X is compact, V^c is compact.

(3) By 7.5.1, $f(V^c)$ is compact in Y, hence closed. Because f is a bijection,

$$f(V) = \big(f(V^c)\big)^c.$$

Therefore $f(V)$ is open in Y. Thus f^{-1} is continuous.

■

[A6] cf. R 4.17; compactness makes continuous images of closed sets closed.

Counterexample 7.6.3

A continuous bijection need not be a homeomorphism

Let

$$\varphi : [0, 2\pi) \to S^1 \subseteq \mathbb{R}^2, \qquad \varphi(t) = (\cos t, \sin t).$$

This map is a continuous bijection onto the unit circle, but $[0, 2\pi)$ is not compact and φ^{-1} is not continuous: points on S^1 approaching $(1, 0)$ from the fourth quadrant have arguments tending to 2π, while $\varphi^{-1}(1, 0) = 0$ (cf. R 4.21).

FAILS: the claim that every continuous bijection is automatically a homeomorphism.

Remark 7.6.4

The compactness engine

The proof of 7.6.2 says that f^{-1} is continuous exactly when f carries open sets to open sets; equivalently, it is enough here that f carry closed sets to closed sets. On a compact domain this is automatic: closed subsets of X are compact, continuous images of compact sets are compact, and compact subsets of a metric space are closed (cf. R 4.17).

Remark 7.6.5

Recognising homeomorphic spaces

The basic homeomorphism questions are: given metric spaces M and N, are they homeomorphic, and what are the continuous maps $M \to N$? Typical comparison examples include the unit circle, a trefoil knot, the perimeter of a square, the 2-torus, $[0,1]$, and the unit disc. Compare the compact-domain homeomorphism theorem R 4.17 and circle example R 4.21.

7.7 Uniform Continuity

Definition 7.7.1

Uniform continuity

A function $f : X \to Y$ is *uniformly continuous* on X (R 4.18) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d_X(p,q) < \delta_\varepsilon \quad \Longrightarrow \quad d_Y\big(f(p), f(q)\big) < \varepsilon.$$

The difference from ordinary continuity is that δ depends only on ε, not on the base point.[5]

Theorem 7.7.2

Compactness gives uniform continuity

If X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous (R 4.19).

Proof 7.7.2

Proof by a Finite Half-Radius Cover.

(1) Let $\varepsilon > 0$. Since f is continuous at each $p \in X$, choose $\delta(p) > 0$ such that

$$d_X(p,q) < \delta(p) \quad \Longrightarrow \quad d_Y\big(f(p), f(q)\big) < \frac{\varepsilon}{2}.$$

(2) Define the half-radius ball

$$J(p) := B^X_{\delta(p)/2}(p).$$

The collection $\{J(p)\}_{p \in X}$ is an open cover of X, so compactness gives $p_1, \dots, p_N$ with

$$X \subseteq J(p_1) \cup \cdots \cup J(p_N).$$

[5] The whole content sits in the order of the quantifiers: a single δ must now work *everywhere at once*, whereas ordinary continuity may choose δ after the point and let it shrink from place to place. Picture sliding a fixed window of width δ along the graph—uniform continuity says one width holds the output swing under ε at every location. The reciprocal $1/x$ fails this near 0, where the graph steepens without bound and the usable δ collapses (7.7.3).

(3) Set

$$\delta = \frac{1}{2}\min(\delta(p_1), \ldots, \delta(p_N)) > 0.$$

This δ depends only on ε.

(4) Take $x, y \in X$ with $d_X(x, y) < \delta$. Since the $J(p_i)$ cover X, choose i with $x \in J(p_i)$. Then

$$d_X(p_i, x) < \frac{1}{2}\delta(p_i),$$

so

$$d_Y\bigl(f(p_i), f(x)\bigr) < \frac{\varepsilon}{2}.$$

(5) Also,

$$d_X(y, p_i) \le d_X(y, x) + d_X(x, p_i) < \delta + \frac{1}{2}\delta(p_i) \le \delta(p_i),$$

so

$$d_Y\bigl(f(y), f(p_i)\bigr) < \frac{\varepsilon}{2}.$$

(6) By the triangle inequality in Y,

$$d_Y\bigl(f(y), f(x)\bigr) \le d_Y\bigl(f(y), f(p_i)\bigr) + d_Y\bigl(f(p_i), f(x)\bigr) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus f is uniformly continuous.

■

[A6+A4] cf. R 4.19; the half-radius leaves room for nearby points.

Counterexample 7.7.3

The reciprocal is not uniformly continuous on $(0, \infty)$

The function $f(x) = 1/x$ is continuous on $(0, \infty)$, but not uniformly continuous there. In the explicit continuity estimate from 7.3.2,

$$\delta(x, \varepsilon) = \min\left(\frac{x}{2}, \frac{\varepsilon x^2}{2}\right) \to 0 \qquad \text{as } x \to 0.$$

Thus no single δ works for all $x > 0$. The domain $(0, \infty)$ is not compact, so 7.7.2 does not apply. On a compact interval $[a, b]$ with $a > 0$, the same function is uniformly continuous. Compare Rudin's noncompact-domain counterexamples in R 4.20.

FAILS: the claim that ordinary continuity on a noncompact domain implies uniform continuity.

Remark 7.7.4

Three compactness-for-free results

Compactness upgrades continuity in three recurring ways: a continuous image of a compact set is compact (7.5.1); a continuous bijection from a compact domain is a homeomorphism (7.6.2); and a continuous map from a compact domain is uniformly continuous (7.7.2). These are R 4.14, R 4.17, and R 4.19.

7.8 Monotonic Functions

For a real function f on an interval, the one-sided limits at an interior point x are written

$$f(x^+) := \lim_{t\to x^+} f(t), \qquad f(x^-) := \lim_{t\to x^-} f(t),$$

when these limits exist; equivalently $f(x^+) = \lim_n f(t_n)$ for every $t_n \to x$ with $t_n > x$, and $f(x^-) = \lim_n f(t_n)$ for every $t_n \to x$ with $t_n < x$ (cf. 7.3.7). The two-sided limit $\lim_{t\to x} f(t)$ exists exactly when $f(x^-)$ and $f(x^+)$ exist and agree. For *monotone* functions these one-sided limits always exist, which forces every discontinuity to be a jump and limits how many there can be (R 4.29–R 4.30).

Definition 7.8.1

Monotonic function

A real function $f : (a, b) \to \mathbb{R}$ is *monotonically increasing* if

$$x < y \quad\Longrightarrow\quad f(x) \le f(y),$$

and *monotonically decreasing* if

$$x < y \quad\Longrightarrow\quad f(x) \ge f(y).$$

A function that is one or the other is called *monotone* (R 4.28). Statements proved for increasing f transfer to decreasing f by replacing f with $-f$.

Proposition 7.8.2

One-sided limits of a monotone function

Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then $f(x^-)$ and $f(x^+)$ exist at every $x \in (a, b)$, and

$$f(x^-) = \sup_{a<t<x} f(t) \;\le\; f(x) \;\le\; \inf_{x<t<b} f(t) = f(x^+).$$

This is Rudin's existence theorem for one-sided limits of monotone functions (R 4.29).

Proof 7.8.2

Proof by a Supremum Argument.

(1) Consider the set $A = \{f(t) : a < t < x\}$. Since f is increasing, $f(t) \le f(x)$ for every $t < x$, so A is bounded above by $f(x)$. Hence

$$\alpha := \sup A$$

exists and $\alpha \le f(x)$.

(2) We show $f(x^-) = \alpha$. Let $\varepsilon > 0$. Since $\alpha - \varepsilon$ is not an upper bound of A, there is

$\delta > 0$ with $a < x - \delta < x$ and

$$\alpha - \varepsilon < f(x - \delta) \leq \alpha.$$

(3) For any $t \in (x - \delta, x)$ monotonicity gives $f(x - \delta) \leq f(t) \leq \alpha$, hence

$$\alpha - \varepsilon < f(t) \leq \alpha, \qquad \text{i.e.} \qquad |f(t) - \alpha| < \varepsilon.$$

Thus $f(x^-) = \alpha = \sup_{a<t<x} f(t)$.

(4) The argument for the right-hand limit is symmetric: $\{f(t) : x < t < b\}$ is bounded below by $f(x)$, its infimum β satisfies $\beta \geq f(x)$, and the same estimate gives $f(x^+) = \beta = \inf_{x<t<b} f(t)$. Combining the two yields $f(x^-) \leq f(x) \leq f(x^+)$.

■

[A5] cf. R 4.29; monotonicity turns the sup into the left-hand limit.

Corollary 7.8.3

Monotone functions have only first-kind discontinuities

If $f : (a, b) \to \mathbb{R}$ is monotone, then every discontinuity of f is of the first kind: both one-sided limits exist, and a discontinuity at x occurs precisely when $f(x^-) \neq f(x^+)$, a jump (R 4.29).

Proof 7.8.3

Proof by Existence of Both One-Sided Limits.

(1) By 7.8.2 (applied to f, or to $-f$ if f is decreasing), the limits $f(x^-)$ and $f(x^+)$ exist at every $x \in (a, b)$. By 7.3.7 this rules out a discontinuity of the second kind.

(2) If $f(x^-) = f(x^+)$ then, since $f(x^-) \leq f(x) \leq f(x^+)$, the common value equals $f(x)$ and f is continuous at x. Hence a discontinuity forces $f(x^-) \neq f(x^+)$, which is a jump.

■

[Direct] cf. R 4.29; immediate from 7.8.2.

Theorem 7.8.4

A monotone function has at most countably many discontinuities

If $f : (a, b) \to \mathbb{R}$ is monotonically increasing, then the set of points at which f is discontinuous is at most countable. The same holds for any monotone f (R 4.30).

Proof 7.8.4

Proof by Injecting a Rational into Each Jump.

(1) Let D be the set of discontinuities. By 7.8.3, each $x \in D$ is a jump, so $f(x^-) < f(x^+)$ and the open interval $\big(f(x^-), f(x^+)\big)$ is nonempty. By density of the rationals, choose a rational $r(x) \in \big(f(x^-), f(x^+)\big)$.

(2) The assignment $x \mapsto r(x)$ is injective. Indeed, if $x_1 < x_2$ are both in D, then for

any t with $x_1 < t < x_2$ monotonicity gives

$$f(x_1^+) = \inf_{s > x_1} f(s) \leq f(t) \leq \sup_{s < x_2} f(s) = f(x_2^-).$$

Hence the jump interval $\big(f(x_1^-), f(x_1^+)\big)$ lies entirely to the left of $\big(f(x_2^-), f(x_2^+)\big)$; the two are disjoint, so $r(x_1) < r(x_2)$.

(3) Thus $r : D \to \mathbb{Q}$ is an injection into a countable set, so D is at most countable.

■

[A5] cf. R 4.30; disjoint jump intervals inject into $\mathbb{Q}$.

Remark 7.8.5

Sharpness: a prescribed countable discontinuity set

The bound "at most countably many" is sharp: for *any* countable set $E = \{x_1, x_2, \dots\} \subseteq (a, b)$ there is a monotone function discontinuous exactly on E. Fix positive reals $c_n > 0$ with $\sum_n c_n$ convergent and set

$$f(x) = \sum_{x_n < x} c_n.$$

Then f is monotonically increasing, with jump $f(x_n^+) - f(x_n^-) = c_n > 0$ at each x_n (so f is discontinuous at every point of E), while f is continuous at every point of E^c. The convergence of $\sum c_n$ keeps f finite and bounded. Compare Rudin's construction of a function with prescribed discontinuities (R 4.31).

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