LECTURE 6

Monotone Convergence, the Squeeze Theorem, and the Upper and Lower Limits

MATH S4061 | Introduction to Modern Analysis I

Thursday, 11 June 2026

Of the small there is no smallest, but always a smaller; and of the large there is always a larger.

ANAXAGORAS, FRAGMENT B3 (5TH C. BCE)

CONTENTS OF THIS LECTURE

Overview. Picking up from completeness, this lecture assembles the basic machinery for limits of real sequences. Monotonicity plus boundedness already forces convergence; the comparison and squeeze principles let limits be read off from neighbours; and for sequences that oscillate, the upper and lower limits lim sup and lim inf record the band the sequence is eventually trapped in, coinciding exactly when the sequence converges. After recording four limits that recur throughout the subject, we open the theory of series: convergence through partial sums, the Cauchy criterion, and the first divergence test. The companion text is Rudin, Chapter 3, especially R 3.12–R 3.40.

6.1 Review of Completeness and Cauchy Sequences

Definition 6.1.1

Complete metric space

A metric space (X, d) is *complete* (R 3.12) if every Cauchy sequence in X converges to a point of X.

Example 6.1.2

Complete and incomplete spaces

- Compact metric spaces are complete (R 3.11).
- $\mathbb{R}^k$ is complete (R 3.11).
- $\mathbb{Q}$ and $(0, 1)$ are *not* complete (cf. R 3.12).

Remark 6.1.3

Testing convergence without the limit

In a complete space — in particular in $\mathbb{R}^k$ — it is enough to check whether a sequence is Cauchy; there is no need to know the limit in advance. In an incomplete space such as $\mathbb{Q}$ this fails: a Cauchy sequence may have no limit (6.1.2; cf. R 3.11–R 3.12).

6.2 Monotone Sequences

Definition 6.2.1

Monotonic sequences

A sequence $\{s_n\}$ is monotonic in Rudin's sense (R 3.13) as follows:

- *monotonic increasing* if $s_n \leq s_{n+1}$ for all n;
- *monotonic decreasing* if $s_n \geq s_{n+1}$ for all n.

"Monotonic" means either increasing or decreasing.

Theorem 6.2.2

Monotone convergence

If $\{s_n\} \subseteq \mathbb{R}$ is monotonic and bounded, then $\{s_n\}$ converges (R 3.14).[1]

Proof 6.2.2

Proof by a Supremum Argument of the Monotone Convergence Theorem.

(1) Treat the increasing case (the decreasing case is analogous). Since $\{s_n\}$ is bounded above, the least-upper-bound property of $\mathbb{R}$ provides

$$s := \sup_n s_n.$$

[1] This is the first place completeness earns its keep by *manufacturing* a limit rather than asking you to guess one. A monotone sequence can head only one way, and boundedness blocks the exit, so the terms have nowhere to go but to pile up against their least upper bound—and that bound, $\sup_n s_n$, *is* the limit. Read the two hypotheses as a division of labour: monotone forbids oscillation, bounded forbids escape to infinity.

(2) Let $\varepsilon > 0$. Because $s - \varepsilon < s$, the number $s - \varepsilon$ is not an upper bound of $\{s_n\}$, so there is N_ε with

$$s_{N_\varepsilon} > s - \varepsilon.$$

(3) As the sequence is increasing and s is an upper bound, for every $n \geq N_\varepsilon$

$$s - \varepsilon < s_{N_\varepsilon} \leq s_n \leq s,$$

hence $|s_n - s| < \varepsilon$. Therefore $s_n \to s$.

■

[A5] cf. R 3.14; the decreasing case follows by applying the increasing case to $\{-s_n\}$.

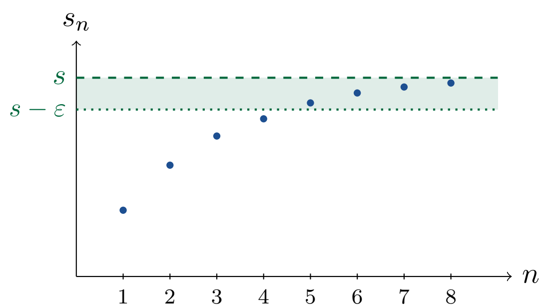

Figure 6.2.3. Climbing to the supremum. An increasing sequence bounded above closes in on $s = \sup_n s_n$: the terms rise but never cross the dashed bound, and from some N_ε on they all lie in the band $(s - \varepsilon, s]$.

6.3 Comparison and Squeeze

Proposition 6.3.1

Order is preserved in the limit

Let $\{s_n\}$ and $\{t_n\}$ be convergent real sequences with $s_n \leq t_n$ for all n. Then

$$\lim_{n\to\infty} s_n \;\leq\; \lim_{n\to\infty} t_n.$$

This is the ordinary-limit form of Rudin's comparison theorem for upper and lower limits (cf. R 3.19).

Idea. Suppose instead that $s := \lim s_n$ and $t := \lim t_n$ satisfy $s > t$. Separate s and t by disjoint ε-balls, with $\varepsilon < \frac{1}{2}|s - t|$, so that $B_\varepsilon(t)$ and $B_\varepsilon(s)$ sit on opposite sides of the midpoint $\frac{s+t}{2}$.

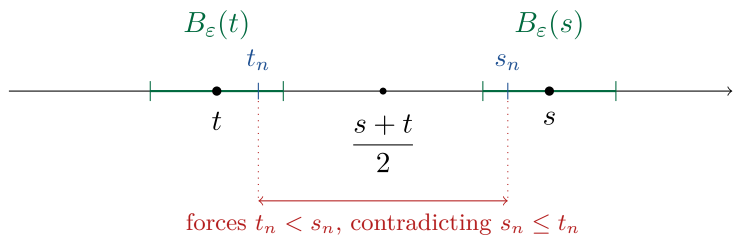

Figure 6.3.2. SEPARATING THE LIMITS. With $\varepsilon < \frac{1}{2}|s-t|$, the balls $B_\varepsilon(t)$ and $B_\varepsilon(s)$ fall on opposite sides of the midpoint $\frac{s+t}{2}$, which is impossible once $s_n \le t_n$.

Proof 6.3.1

Proof by Contradiction that Order Passes to the Limit.

(1) Write $s = \lim s_n$, $t = \lim t_n$, and suppose for contradiction $s > t$. Since $s - t > 0$, choose

$$0 < \varepsilon < \tfrac{1}{2}|s-t| = \tfrac{1}{2}(s-t).$$

(2) By convergence there are N_1, N_2 with $|s_n - s| < \varepsilon$ for $n \ge N_1$ and $|t_n - t| < \varepsilon$ for $n \ge N_2$. With $N = \max\{N_1, N_2\}$, for every $n \ge N$

$$s_n > s - \varepsilon \qquad \text{and} \qquad t_n < t + \varepsilon.$$

(3) The choice of ε puts these on opposite sides of the midpoint:

$$t + \varepsilon < t + \tfrac{1}{2}(s-t) = \frac{s+t}{2} = s - \tfrac{1}{2}(s-t) < s - \varepsilon.$$

(4) Chaining the inequalities, for every $n \ge N$,

$$t_n < t + \varepsilon < \frac{s+t}{2} < s - \varepsilon < s_n,$$

so $t_n < s_n$, contradicting $s_n \le t_n$. Hence $s \le t$, i.e. $\lim s_n \le \lim t_n$. ∎

[A9]

Proposition 6.3.3

Squeeze theorem

Suppose $r_n \le s_n \le t_n$ for all n, where $r_n \to \ell$ and $t_n \to \ell$. Then $s_n \to \ell$. This is the comparison principle used in Rudin's special-sequence computations (cf. R 3.20).

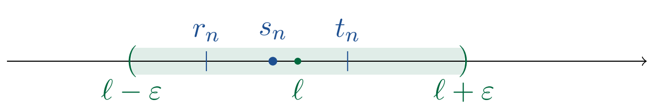

Figure 6.3.4. The squeeze. Once r_n and t_n lie in $(\ell-\varepsilon, \ell+\varepsilon)$, the trapped term s_n must lie there too.

Proof 6.3.3

Proof by an ε–N Estimate of the Squeeze Theorem.

(1) Let $\varepsilon > 0$. Since $r_n \to \ell$ and $t_n \to \ell$, there is N with, for all $n \geq N$,

$$|r_n - \ell| < \varepsilon \qquad \text{and} \qquad |t_n - \ell| < \varepsilon,$$

so $r_n, t_n \in (\ell-\varepsilon, \ell+\varepsilon)$.

(2) Using $r_n \leq s_n \leq t_n$, for every $n \geq N$

$$\ell - \varepsilon < r_n \leq s_n \leq t_n < \ell + \varepsilon,$$

hence $|s_n - \ell| < \varepsilon$. As $\varepsilon > 0$ was arbitrary, $s_n \to \ell$. ■

[A2]

6.4 The Upper and Lower Limits

Convergence $s_n \to \ell$ says that for every $\varepsilon > 0$ there is N with $s_n \in (\ell-\varepsilon, \ell+\varepsilon)$ for $n \geq N$. For a bounded sequence that does *not* converge, no single ℓ works. Instead there are two values $\ell^- \leq \ell^+$ such that the sequence is eventually trapped in any ε-enlargement of the band $[\ell^-, \ell^+]$:

$$\forall\, \varepsilon > 0 \;\; \exists N \;\; \text{s.t.} \;\; s_n \in (\ell^- - \varepsilon,\; \ell^+ + \varepsilon) \quad \text{for } n \geq N.$$

We write $\ell^- = \liminf s_n$ and $\ell^+ = \limsup s_n$.

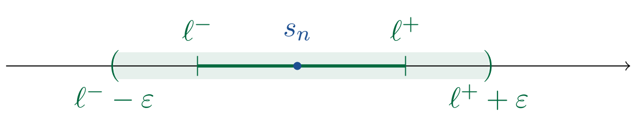

Figure 6.4.1. The trapping band. For a bounded sequence, every ε-enlargement of $[\ell^-, \ell^+]$ eventually contains all the terms.

The two coincide exactly when the sequence converges:

$$s_n \to \ell \iff \liminf s_n = \limsup s_n = \ell.$$

Example 6.4.2

An oscillating sequence

For $s_n = (-1)^n \dfrac{n}{n-1}$ (with $n \geq 2$), the even terms approach $+1$ from above and the odd terms approach -1 from below. Hence $\liminf s_n = -1$ and $\limsup s_n = +1$, and the sequence does not converge, since these differ (cf. R 3.18).

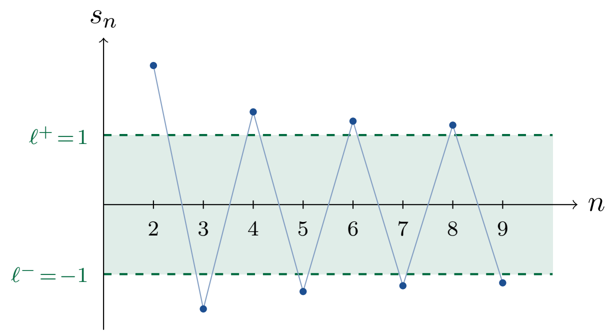

Figure 6.4.3. TWO LIMIT POINTS. The oscillating sequence $s_n = (-1)^n \frac{n}{n-1}$ has no limit: the even terms fall toward $\ell^+ = \limsup s_n = 1$ from above and the odd terms rise toward $\ell^- = \liminf s_n = -1$ from below. Every ε-enlargement of the shaded band $[\ell^-, \ell^+]$ eventually traps the whole tail.

Definition 6.4.4

Divergence to $\pm\infty$

For a real sequence $\{t_n\}$ (R 3.15),

- $t_n \to +\infty$ means: for every $M > 0$ there is N with $t_n > M$ for all $n \geq N$;
- $t_n \to -\infty$ means: for every $M > 0$ there is N with $t_n < -M$ for all $n \geq N$.

Such sequences diverge, but in a specific, controlled way.

Definition 6.4.5

Limit inferior and limit superior

The *upper* and *lower limits* of a real sequence $\{s_n\}$ are defined by the tail extrema

$$\limsup s_n = \lim_{n\to\infty} \Big(\sup_{k\geq n} s_k\Big), \qquad \liminf s_n = \lim_{n\to\infty} \Big(\inf_{k\geq n} s_k\Big).$$

The tails are nested, $\{s_k : k \geq n+1\} \subseteq \{s_k : k \geq n\}$, so $\sup_{k\geq n} s_k$ is non-increasing in n (and $\inf_{k\geq n} s_k$ non-decreasing); hence both limits always exist in the extended reals $[-\infty, +\infty]$.[2] Two characterizations follow.

(1) Subsequential limits. If $\{s_n\}$ is bounded and E is its set of subsequential limits, then

$$\liminf s_n = \min E, \qquad \limsup s_n = \max E.$$

[2]Picture it this way: $\sup_{k\geq n} s_k$ is the ceiling over the tail from n onward, and discarding more early terms can only lower a supremum, so these ceilings march downward to a limit—the lim sup. It is the highest value the sequence keeps crowding back up to, while lim inf is the matching floor. A convergent sequence has ceiling and floor equal; a bounded oscillating one holds them apart, pinning the tail between two honest numbers (6.4.2).

(2) Eventual trapping (finite $\liminf, \limsup$). For every $\varepsilon > 0$ there is N_ε with

$$s_n \in \big(\liminf s_n - \varepsilon,\ \limsup s_n + \varepsilon\big) \qquad \text{for } n \geq N_\varepsilon,$$

and $[\liminf s_n,\ \limsup s_n]$ is the smallest closed interval with this property.

This tail-extrema presentation is equivalent to Rudin's subsequential-limit definition (R 3.16) and characterization (R 3.17).

Remark 6.4.6

The extreme limits are attained

For a *bounded* sequence, E is closed (6.4.8), bounded, and nonempty, so it contains its own extrema: $\inf E \in E$ and $\sup E \in E$. Hence $\liminf s_n$ and $\limsup s_n$ are themselves subsequential limits of $\{s_n\}$. (For an unbounded sequence this holds in the extended reals, with $\pm\infty$ admitted as subsequential limits.) Compare R 3.17.

Remark 6.4.7

Existence

If $\{s_n\}$ is bounded, then E is bounded and nonempty, so $\liminf s_n$ and $\limsup s_n$ exist as finite reals. If $\{s_n\}$ is unbounded, then E may include $+\infty$ and/or $-\infty$, and $\liminf s_n, \limsup s_n$ may equal $\pm\infty$ (cf. R 3.16–R 3.17).

Proposition 6.4.8

The set of subsequential limits is closed

Let $\{p_n\}$ be a sequence in a metric space (X, d), and let $E^* \subseteq X$ be its set of subsequential limits. Then E^* is a closed subset of X (R 3.7). In particular, for a bounded real sequence the set E of 6.4.5–6.4.6 is closed, justifying that it contains its extrema.

Proof 6.4.8

Proof by Diagonal Extraction that E^* is Closed.

(1) Let q be a cluster point of E^*; we construct a subsequence of $\{p_n\}$ converging to q, which shows $q \in E^*$ and hence that E^* is closed. Set $\delta = d(q, p_{n_1})$ where $p_{n_1} \neq q$ (if $\{p_n\}$ is eventually constant at q, then $q \in E^*$ trivially).

(2) Suppose $n_1, \ldots, n_{k-1}$ have been chosen. Since q is a cluster point of E^*, pick $x \in E^*$ with $d(x, q) < 2^{-k}\delta$. As $x \in E^*$, some subsequence of $\{p_n\}$ converges to x, so we may choose $n_k > n_{k-1}$ with $d(p_{n_k}, x) < 2^{-k}\delta$.

(3) By the triangle inequality,

$$d(p_{n_k}, q) \leq d(p_{n_k}, x) + d(x, q) < 2^{-k}\delta + 2^{-k}\delta = 2^{1-k}\delta \xrightarrow[k\to\infty]{} 0.$$

Hence $p_{n_k} \to q$, so $q \in E^*$. Therefore E^* contains all its cluster points and is closed.

■

[A8] cf. R 3.7; build a subsequence converging to a given cluster point of E^*.

6.5 Some Standard Limits

Proposition 6.5.1

Four standard limits

(a) If $|x| < 1$, then $x^n \to 0$.

(b) If $p > 0$, then $\dfrac{1}{n^p} \to 0$.

(c) If $p > 1$, then $\sqrt[n]{p} \to 1$.

(d) $\sqrt[n]{n} \to 1$.

These are the recurring special limits collected in R 3.20(a)–(c),(e).

Proof 6.5.1.a

Proof by the Binomial Inequality of $x^n \to 0$.

(1) If $x = 0$ then $x^n = 0 \to 0$, so assume $0 < |x| < 1$. Then $1/|x| > 1$, so $|x| = \dfrac{1}{1+p}$ for a unique $p > 0$ (the map $p \mapsto \frac{1}{1+p}$ is a bijection $(0, \infty) \to (0, 1)$).

(2) By the binomial theorem, keeping the nonnegative $k = 0, 1$ terms,

$$(1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k \;\geq\; 1 + np.$$

(3) Hence

$$|x|^n = \frac{1}{(1+p)^n} \leq \frac{1}{1+np} \xrightarrow[n\to\infty]{} 0,$$

and since $|x^n| = |x|^n \to 0$, we conclude $x^n \to 0$.

■

[A2]

Proof 6.5.1.b

Proof by an ε–N Estimate of $1/n^p \to 0$.

(1) Let $\varepsilon > 0$ and choose N with $N > (1/\varepsilon)^{1/p}$, equivalently $\dfrac{1}{N^p} < \varepsilon$.

(2) Then for every $n \geq N$,

$$\frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon,$$

so $\dfrac{1}{n^p} \to 0$.

■

[A2]

Proof 6.5.1.c

Proof by the Squeeze Theorem of $\sqrt[n]{p} \to 1$.

(1) Assume $p > 1$ and set $x_n = \sqrt[n]{p} - 1 \geq 0$. By the binomial inequality from (a),

$$1 + nx_n \leq (1 + x_n)^n = p,$$

so $0 \leq x_n \leq \dfrac{p-1}{n}$.

(2) Since $\dfrac{p-1}{n} \to 0$, the squeeze theorem (6.3.3) gives $x_n \to 0$, i.e. $\sqrt[n]{p} = 1 + x_n \to 1$. ∎

[A2] uses 6.3.3.

Proof 6.5.1.d

Proof by the Squeeze Theorem of $\sqrt[n]{n} \to 1$.

(1) Set $x_n = \sqrt[n]{n} - 1 \geq 0$, so $n = (1 + x_n)^n$. The linear term is too weak, so keep the $k = 2$ term:

$$n = (1 + x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2 \qquad (n \geq 2).$$

(2) Rearranging, $x_n^2 \leq \dfrac{2}{n-1}$, hence $0 \leq x_n \leq \sqrt{\dfrac{2}{n-1}}$.

(3) Since $\sqrt{\frac{2}{n-1}} \to 0$, the squeeze theorem (6.3.3) gives $x_n \to 0$, i.e. $\sqrt[n]{n} = 1 + x_n \to 1$. ∎

[A2] keep the $k = 2$ binomial term; uses 6.3.3.

Remark 6.5.2

The case $0 < p < 1$ *in (c)*

For $0 < p < 1$, apply part (c) to $\sigma = 1/p > 1$: since $\sqrt[n]{\sigma} \to 1$, also $\sqrt[n]{p} = 1/\sqrt[n]{\sigma} \to 1$ (and $\sqrt[n]{1} = 1$). Hence $\sqrt[n]{p} \to 1$ for every $p > 0$; the full $p > 0$ statement is R 3.20(b).

Proposition 6.5.3

Polynomial versus geometric growth

If $p > 0$ and $\alpha \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0.$$

Powers of n grow more slowly than any fixed exponential base $1 + p > 1$; this is R 3.20(d).[3]

Proof 6.5.3

Proof by the Binomial Inequality and Squeeze of $n^\alpha/(1+p)^n \to 0$.

[3] The single linear or quadratic binomial term used for (a) and (d) above is no longer enough: against n^α one must keep a binomial term of degree higher than α. Keeping the k-th term with $k > \alpha$ buries the numerator's n^α under an n^k in the denominator, and the squeeze finishes the job.

(1) Choose an integer $k > \alpha$ with $k > 0$. For $n > 2k$ the binomial theorem, keeping only the nonnegative k-th term, gives

$$(1+p)^n \geq \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!}\, p^k > \left(\frac{n}{2}\right)^k \frac{p^k}{k!},$$

since each of the k factors $n, n-1, \ldots, n-k+1$ exceeds $n/2$ once $n > 2k$.

(2) Inverting and multiplying by $n^\alpha > 0$,

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k}\, n^{\alpha-k} \qquad (n > 2k).$$

(3) Since $\alpha - k < 0$, write $n^{\alpha-k} = 1/n^{k-\alpha}$ with $k - \alpha > 0$; by 6.5.1(b) this tends to 0, so the right-hand bound tends to 0. The squeeze theorem (6.3.3) then gives $\dfrac{n^\alpha}{(1+p)^n} \to 0$.

■

[A2] cf. R 3.20(d); keep the k-th binomial term with $k > \alpha$; uses 6.3.3 and 6.5.1(b).

6.6 Introduction to Series

(Not covered on the midterm.)

As a field, $\mathbb{R}$ hands us directly only the polynomials $1, x, x^2, \ldots$ and, as their quotients, the rational functions $P(x)/Q(x)$. What about e^x, $\sin x$, $\cos x$? These are reached not by field operations but by infinite sums — for instance

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

To make sense of such a sum we must first say what an infinite sum *is*.

Definition 6.6.1

Series and partial sums

Given a sequence $\{a_n\}$ in $\mathbb{R}$, its *k-th partial sum* is

$$s_k = a_1 + \cdots + a_k = \sum_{i=1}^{k} a_i \qquad (s_0 = 0,\ \text{the empty sum}).$$

If $s_k \to s$, the *series* $\sum a_n$ *converges*, and we write

$$\sum_{i=1}^{\infty} a_i = \lim_{k\to\infty} s_k = s.$$

A series may instead be indexed from any starting point—often $n = 0$, as for $\sum c_n x^n$—with the partial sums adjusted accordingly.[4] This is Rudin's series convention (R 3.21) with the real-valued specialization used in lecture.

Theorem 6.6.2

Cauchy criterion for series

A series $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is N such that

$$\Big|\sum_{k=m}^{n} a_k\Big| < \varepsilon \qquad \text{for all } n \geq m \geq N.$$

This is the Cauchy criterion for series (R 3.22).

Proof 6.6.2

Proof by Completeness of $\mathbb{R}$ of the Cauchy Criterion.

(1) For $n \geq m \geq 1$ one has $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$ (with $s_0 = 0$), so the displayed condition says precisely that the partial sums $\{s_k\}$ form a Cauchy sequence.

(2) Since $\mathbb{R}$ is complete (6.1.1), $\{s_k\}$ converges if and only if it is Cauchy. Hence $\sum a_n$ converges if and only if the criterion holds.

■

[Direct] cf. R 3.22; the partial sums converge iff they are Cauchy, and $\mathbb{R}$ is complete.

Corollary 6.6.3

Vanishing of the terms

If $\sum a_n$ converges, then $a_n \to 0$ (R 3.23).[5]

Proof 6.6.3

Proof by the Cauchy Criterion with $m = n$.

(1) Take $m = n$ in the criterion (6.6.2): for every $\varepsilon > 0$ there is N with

$$|a_n| = \Big|\sum_{k=n}^{n} a_k\Big| < \varepsilon \qquad \text{for all } n \geq N.$$

(2) Therefore $a_n \to 0$.

■

[Direct] cf. R 3.23.

So $a_n \to 0$ is *necessary* for convergence. Is it *sufficient*? No.

Proposition 6.6.4

[4] A series introduces no genuinely new operation; there is no act of adding infinitely many numbers at once. The symbol $\sum a_n$ is shorthand for a *sequence*—the partial sums s_k—together with its limit, so every question about a series is a question about that sequence, and every tool for sequences (monotonicity, the Cauchy criterion, the algebra of limits) carries straight over.

[5] Use this in one direction only. It says convergence *forces* the terms to die out, so if $a_n \not\to 0$ the series cannot converge—a quick divergence test. It does not say the reverse: terms dying out is not enough, as the harmonic series shows next (6.6.4). The terms must not merely vanish but vanish fast enough to be summable.

The harmonic series diverges

Although $\dfrac{1}{n} \to 0$, the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This is the $p = 1$ case of Rudin's p-series theorem (R 3.28).

Proof 6.6.4

Proof by Dyadic Grouping of the Divergence of the Harmonic Series.

(1) Round each term down to the next power of $\frac{1}{2}$: replace $\frac{1}{3}$ by $\frac{1}{4}$; then $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ by $\frac{1}{8}$; and so on. If s'_k denotes the resulting partial sums, then $s_k \geq s'_k$ for every k.

(2) The rounded series groups into dyadic blocks of equal weight:

$$1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots ,$$

each block summing to $\frac{1}{2}$; hence $s'_{2^m} = 1 + \frac{m}{2}$.

(3) Therefore $s_{2^m} \geq s'_{2^m} = 1 + \frac{m}{2} \to \infty$, so the partial sums $\{s_n\}$ have a subsequence diverging to $+\infty$ and cannot converge. Thus $\sum \frac{1}{n}$ diverges.

■

[Direct] Cauchy condensation; cf. R 3.27–R 3.28.

Proposition 6.6.5

The p-series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$. Rudin proves the full convergence/divergence dichotomy in R 3.28.

Proof 6.6.5

Proof of the Convergence of the p-Series.

↪ Stated in lecture as a fact, for use as a comparison series; cf. R 3.28.

Proposition 6.6.6

A nonnegative series converges when bounded

A series $\sum a_n$ of nonnegative terms ($a_n \geq 0$) converges if and only if its partial sums are bounded (R 3.24).

Proof 6.6.6

Proof by Monotone Convergence for a Series of Nonnegative Terms.

(1) Since $a_{n+1} \geq 0$, the partial sums satisfy $s_{n+1} = s_n + a_{n+1} \geq s_n$, so $\{s_n\}$ is increasing.

(2) If the partial sums are bounded, then $\{s_n\}$ is increasing and bounded, hence converges by the monotone convergence theorem (6.2.2); that is, $\sum a_n$ converges. Conversely, a convergent sequence is bounded.

■

[A5] cf. R 3.24; the partial sums increase; apply 6.2.2.

Proposition 6.6.7

The geometric series

For $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

This is the convergent half of Rudin's geometric-series theorem (R 3.26).

Proof 6.6.7

Proof by the Partial-Sum Formula of the Geometric Series.

(1) For $x \neq 1$ the partial sums have the closed form

$$s_k = 1 + x + \cdots + x^k = \frac{1-x^{k+1}}{1-x}.$$

(2) Since $|x| < 1$ gives $|x|^{k+1} \to 0$, we have $x^{k+1} \to 0$ and hence

$$s_k \xrightarrow[k\to\infty]{} \frac{1}{1-x},$$

so $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

∎

[Direct] cf. R 3.26; uses $|x|^k \to 0$ for $|x| < 1$ (6.5.1).

Theorem 6.6.8

Comparison test for series

(a) If $|a_n| \leq c_n$ for all $n \geq N_0$ and $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for all n and $\sum d_n$ diverges, then $\sum a_n$ diverges.

This is Rudin's comparison test (R 3.25).[6]

Proof 6.6.8

Proof by the Cauchy Criterion of the Comparison Test.

(1) For (a), let $\varepsilon > 0$. Since $\sum c_n$ converges, the Cauchy criterion (6.6.2) gives some $N \geq N_0$ with $\sum_{k=m}^{n} c_k < \varepsilon$ for all $n \geq m \geq N$. Then for such n, m,

$$\Big|\sum_{k=m}^{n} a_k\Big| \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} c_k < \varepsilon,$$

so $\sum a_n$ meets the Cauchy criterion and therefore converges.

[6] Convergence of a series is really a question of how fast the terms shrink, and comparison is the basic way to settle it: outrun a series known to converge and you converge too; be outrun by one known to diverge and you diverge. Most of the later tests are this one in disguise—they simply pick a convenient series, usually geometric, to measure against.

(2) For (b), suppose instead $\sum a_n$ converged. Since $0 \le d_n \le a_n = |a_n|$, part (a) (with $c_n = a_n$) would force $\sum d_n$ to converge — contradicting that it diverges. Hence $\sum a_n$ diverges.

■

[A4] cf. R 3.25; bound the tail of $\sum a_n$ by that of $\sum c_n$.

The natural series to compare against is the geometric one. Comparing $\sum a_n$ to a geometric $\sum b_n$: if the ratio b_{n+1}/b_n is constant we are led to the *ratio test*, and if the root $\sqrt[n]{b_n}$ is constant, to the *root test*.

Theorem 6.6.9

Root test

Given $\sum a_n$, set $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

- if $\alpha < 1$, $\sum a_n$ converges;
- if $\alpha > 1$, $\sum a_n$ diverges;
- if $\alpha = 1$, the test is inconclusive.

This is Rudin's root test (R 3.33).

Proof 6.6.9

Proof by Comparison with a Geometric Series of the Root Test.

(1) If $\alpha < 1$, choose β with $\alpha < \beta < 1$. Then $\sqrt[n]{|a_n|} < \beta$ for all large n, so $|a_n| < \beta^n$; as $\sum \beta^n$ converges, the comparison test (6.6.8) gives convergence.

(2) If $\alpha > 1$, then $\sqrt[n]{|a_n|} > 1$ for infinitely many n, so $|a_n| > 1$ infinitely often and $a_n \not\to 0$; by 6.6.3 the series diverges. For $\alpha = 1$ the test decides nothing: both $\sum \frac{1}{n}$ (divergent) and $\sum \frac{1}{n^2}$ (convergent) have $\alpha = 1$.

■

[A1] cf. R 3.33; uses the comparison test (6.6.8).

Theorem 6.6.10

Ratio test

Suppose $a_n \neq 0$ for all large n. Then $\sum a_n$ converges if $\limsup\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, and diverges if there is N with $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ for all $n \ge N$. This is Rudin's ratio test (R 3.34).

Proof 6.6.10

Proof by Comparison with a Geometric Series of the Ratio Test.

(1) If $\limsup |a_{n+1}/a_n| < 1$, choose β in between; then $|a_{n+1}/a_n| < \beta$ for $n \ge N$, so $|a_n| \le |a_N|\,\beta^{n-N}$ for $n \ge N$. Comparison with the geometric series (6.6.8) gives convergence.

(2) If $|a_{n+1}/a_n| \ge 1$ for all $n \ge N$, then $|a_n| \ge |a_N| > 0$ for $n \ge N$, so $a_n \not\to 0$ and the series diverges (6.6.3).

■

[A1] cf. R 3.34; uses the comparison test (6.6.8).

Definition 6.6.11

Power series and radius of convergence

A *power series* is a function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

Fix $x \neq 0$ and apply the root test (6.6.9)[7] to $\sum c_n x^n$. With $\alpha := \limsup_{n\to\infty} \sqrt[n]{|c_n|}$,

$$\beta := \limsup_{n\to\infty} \sqrt[n]{|c_n x^n|} = |x|\,\alpha,$$

so the series converges when $\beta = |x|\,\alpha < 1$ and diverges when $|x|\,\alpha > 1$. Hence it converges for $|x| < R$, where

$$R := \frac{1}{\alpha} \qquad \big(R := +\infty \text{ if } \alpha = 0, \qquad R := 0 \text{ if } \alpha = +\infty\big).$$

R is the *radius of convergence*: the series converges on $(-R, R)$ and diverges for $|x| > R$; at $x = 0$ it is just $f(0) = c_0$, and the endpoints $|x| = R$ require separate analysis. This is Rudin's power-series setup and radius formula (R 3.38–R 3.39), restricted here to real x.

Example 6.6.12

The exponential series

For $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ the ratios have a limit, so they compute α (the valid case for the ratio shortcut):

$$\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 = \alpha,$$

so $R = 1/\alpha = +\infty$: the series converges for every x. This function is $f(x) = e^x$. Compare Rudin's examples following the radius-of-convergence theorem (R 3.40).

Remark 6.6.13

Analytic $\Rightarrow C^\infty$, but not conversely

A function defined by a power series on $(-R, R)$ is called *analytic* there. Such a function may be differentiated termwise infinitely often, and each derivative is again given by a power series with the same radius of convergence—so an analytic function is automatically C^∞, i.e. smooth. The converse, however, fails: there are functions that are infinitely differentiable everywhere yet are *not* analytic, because their Taylor series, though convergent, does not return the function. Being analytic is thus strictly stronger than being smooth. A standard such example—e^{-1/x^2} extended by $f(0) = 0$, whose derivatives at 0 all vanish—appears later (8.7.2). The termwise differentiation of power series will be proved when we treat uniform convergence.

[7] The root test is used here because it is strictly more general than the ratio test; for the precise relationship see the appendix, §6.A.1.

6.7 Absolute Convergence and Rearrangements

Adding two convergent series is harmless: $(\sum a_n) + (\sum b_n) = \sum(a_n + b_n)$. Multiplying or *reordering* the terms of a series is far more delicate and demands a stronger hypothesis than mere convergence. The right hypothesis is that the series of absolute values converge.

Definition 6.7.1

Absolute convergence

A series $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges (R 3.45). A series that converges but does not converge absolutely is called *conditionally convergent*.

Proposition 6.7.2

Absolute convergence implies convergence

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges (R 3.45).[8]

Proof 6.7.2

Proof by the Cauchy Criterion that Absolute Convergence Implies Convergence.

(1) Let $\varepsilon > 0$. Since $\sum |a_n|$ converges, the Cauchy criterion (6.6.2) provides N with $\sum_{k=m}^{n} |a_k| < \varepsilon$ for all $n \geq m \geq N$. Then for such n, m,

$$\Big|\sum_{k=m}^{n} a_k\Big| \leq \sum_{k=m}^{n} |a_k| < \varepsilon.$$

(2) Thus $\sum a_n$ satisfies the Cauchy criterion (6.6.2), and therefore converges.

■

[A4] cf. R 3.45; the triangle inequality reduces the tail of $\sum a_n$ to that of $\sum |a_n|$; uses 6.6.2.

Definition 6.7.3

Rearrangement

Let $\{k_n\}$ be a sequence of positive integers in which every positive integer appears exactly once (a bijection of $\mathbb{N}$). Put $a'_n = a_{k_n}$. The series $\sum a'_n$ is called a *rearrangement* of $\sum a_n$ (R 3.52); it has exactly the same terms, reordered.

Theorem 6.7.4

Riemann's rearrangement theorem

Suppose $\sum a_n$ converges but does not converge absolutely. Let $-\infty \leq \alpha \leq \beta \leq +\infty$ be given. Then there is a rearrangement $\sum a'_n$, with partial sums s'_n, such that

$$\liminf_{n\to\infty} s'_n = \alpha \qquad \text{and} \qquad \limsup_{n\to\infty} s'_n = \beta.$$

[8] The converse is false. The alternating harmonic series $\sum(-1)^{n+1}/n$ converges, yet the series of absolute values is the harmonic series (6.6.4), which diverges. Absolute convergence is the genuinely stronger property, and—as the next theorem shows—the only one robust enough to survive reordering.

In particular, a conditionally convergent series can be rearranged to converge to *any* prescribed sum (take $\alpha = \beta$), or to diverge to $\pm\infty$. This is R 3.54.[9]

Proof 6.7.4

Proof Sketch by Alternately Overshooting β and Undershooting α.

(1) **The two parts each diverge.** Write $p_n = \max\{a_n, 0\}$ and $q_n = \max\{-a_n, 0\}$ for the positive and negative parts, so $a_n = p_n - q_n$ and $|a_n| = p_n + q_n$. If both $\sum p_n$ and $\sum q_n$ converged, then $\sum |a_n| = \sum(p_n + q_n)$ would converge, contradicting non-absolute convergence; if exactly one converged, then $\sum a_n = \sum(p_n - q_n)$ would diverge, contradicting convergence. Hence *both* $\sum p_n$ and $\sum q_n$ diverge to $+\infty$. Also $a_n \to 0$ (6.6.3), so $p_n \to 0$ and $q_n \to 0$.

(2) **Alternately overshoot and undershoot.** Take the positive terms in order and add just enough of them to push the partial sum strictly above β; this is possible because $\sum p_n = +\infty$. Then take negative terms in order and add just enough to pull the partial sum strictly below α; this is possible because $\sum q_n = +\infty$. Repeat indefinitely. Every term is used exactly once (each part is exhausted in order), so the result is a genuine rearrangement.

(3) **The overshoots vanish.** At each switch the partial sum crosses β (resp. α) by no more than the last term added, and since $p_n, q_n \to 0$ these overshoots tend to 0. Therefore the partial sums s'_n cluster down to α and up to β with no other limit points, giving $\liminf s'_n = \alpha$ and $\limsup s'_n = \beta$. When $\alpha = \beta$ (finite) this forces $s'_n \to \alpha$; when $\alpha = \beta = \pm\infty$ the same scheme drives $s'_n \to \pm\infty$.

■

[A9] sketch, following R 3.54; positive and negative parts each diverge while the terms vanish.

[9] This is the cautionary tale of the subject: for a merely conditionally convergent series the value of the sum is an artifact of the *order* of summation, not of the terms alone. Absolute convergence (6.7.1) is precisely what immunizes a series against this—every rearrangement of an absolutely convergent series converges, and to the same sum (R 3.55).

Appendix

Supplementary material—additional proofs, context, and other treatments; not part of the lecture proper.

Supplement 6.A.1

The root test subsumes the ratio test

For any sequence $\{c_n\}$ with $c_n \neq 0$,

$$\liminf_n \left|\frac{c_{n+1}}{c_n}\right| \;\le\; \liminf_n \sqrt[n]{|c_n|} \;\le\; \limsup_n \sqrt[n]{|c_n|} \;\le\; \limsup_n \left|\frac{c_{n+1}}{c_n}\right|$$

(R 3.37). Hence whenever the ratio test settles convergence the root test settles it too, but not conversely; and *if* $\lim_n |c_{n+1}/c_n| = L$ exists then the root quantity tends to L as well, so the ratio test gives the same radius $R = 1/L$ (the converse fails — the radii can agree without the ratio limit existing). This is why 6.6.11 fixes R through the root quantity $\alpha = \limsup_n \sqrt[n]{|c_n|}$.

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