LECTURE 5

Sequences: Convergence, Cauchy Completeness, and $\mathbb{R}^k$

MATH S4061 | Introduction to Modern Analysis I

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A mathematician who is not also something of a poet will never be a complete mathematician.

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Contents of this lecture

Overview. A sequence converges to p when it eventually lies in every ball about p. Two things must hold: the terms must settle, *and* a limit must exist in the space—the second can fail. We develop the calculus of limits (the open-set view, uniqueness, boundedness, the algebra of limits in $\mathbb{R}/\mathbb{C}$, coordinatewise convergence in $\mathbb{R}^k$), then *subsequences*—every sequence in a compact space has a convergent one—and finally *Cauchy sequences* and *completeness*: a Cauchy sequence is one whose terms bunch together, every convergent sequence is Cauchy, and a space is *complete* when the converse holds. Compact spaces and $\mathbb{R}^k$ are complete.

5.1 Convergence in a Metric Space

Definition 5.1.1

Convergent sequence

A sequence $\{p_n\}$ in (X, d) (2.1.6)[1] *converges* (R 3.1) to $p \in X$ if for every $\varepsilon > 0$ there is an N with $d(p_n, p) < \varepsilon$ for all $n \geq N$. We write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$, and call p the *limit*.[2]

Remark 5.1.2

Two things must happen

Convergence asks both that the terms settle *and* that a limit exist *in* X. The latter can fail: in $X = (0, \infty)$ with the usual metric, $p_n = \frac{1}{n}$ has terms that settle, yet diverges—its would-be limit 0 is not in X (cf. R 3.1, R 3.12).

Proposition 5.1.3

Basic properties of limits

Let $\{p_n\}$ be a sequence in (X, d).

1. $p_n \to p \iff$ every open set $U \ni p$ contains p_n for all but finitely many n;
2. limits are unique: if $p_n \to p$ and $p_n \to p'$ then $p = p'$;
3. if $\{p_n\}$ converges then it is bounded;
4. if p is a cluster point of $E \subseteq X$, some sequence in $E \setminus \{p\}$ converges to p.

Proof 5.1.3

Direct Proof of the Four Properties.

(1) For the open-set criterion, if $p_n \to p$ and $U \ni p$ is open, some $B_r(p) \subseteq U$; past the N for $\varepsilon = r$, $p_n \in B_r(p) \subseteq U$. Conversely, $B_\varepsilon(p)$ is open and contains p, so it holds p_n for all but finitely many n; past those, $d(p_n, p) < \varepsilon$.

(2) For uniqueness, given $\varepsilon > 0$, $d(p, p') \leq d(p, p_n) + d(p_n, p') < 2\varepsilon$ for large n. As this holds for *every* $\varepsilon > 0$, $d(p, p') = 0$, so $p = p'$.

(3) For boundedness, take N with $d(p_n, p) < 1$ for $n \geq N$ ($\varepsilon = 1$). With $r = 1 + \max\{0, d(p_1, p), \ldots, d(p_{N-1}, p)\}$, every $p_n \in B_r(p)$, so $\{p_n\}$ is bounded.

(4) For the cluster-point claim, pick for each n a point $p_n \in B_{1/n}(p) \cap E$ with $p_n \neq p$ (possible since p is a cluster point). Then $d(p_n, p) < \frac{1}{n}$, so $p_n \to p$.

■

[Direct] cf. R 3.2. Uniqueness needs no Archimedean property: $d(p, p') < 2\varepsilon$ for all $\varepsilon > 0$ already forces $d(p, p') = 0$.

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[1] We index from $n = 1$ here (Rudin's convention), so $\frac{1}{n}$ is always defined; the sequence definition 2.1.6 permits $n = 0$, which changes nothing.

[2] Read the two quantifiers as a challenge and a response: you name a tolerance ε—a ball $B_\varepsilon(p)$, however small—and the sequence answers with a stage N past which *every* later term sits inside that ball. Convergence is the promise that any challenge can be met: each ball about p, no matter how tight, eventually traps the whole tail. The order is what bites—ε comes first, and N may depend on it.

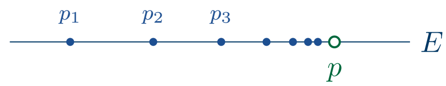

Figure 5.1.4. A cluster point is a limit of points of E. Choosing $p_n \in B_{1/n}(p) \cap E$ with $p_n \neq p$ produces a sequence in E converging to the cluster point p (5.1.3(d)).

5.2 The Algebra of Limits and Convergence in $\mathbb{R}^k$

Proposition 5.2.1

The algebra of limits

Let $s_n \to s$ and $t_n \to t$ in $\mathbb{R}$ (or $\mathbb{C}$). Then $s_n + t_n \to s + t$; $cs_n \to cs$ for any constant c; $s_n t_n \to st$; and $1/s_n \to 1/s$ provided $s_n, s \neq 0$.

Proof 5.2.1

Proof of the Algebra of Limits by Adding and Subtracting Cross Terms.

(1) For the sum, $|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ once both terms are $< \frac{\varepsilon}{2}$.

(2) For a scalar $c \neq 0$, $|cs_n - cs| = |c|\,|s_n - s| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon$; the case $c = 0$ is trivial.

(3) For the product, writing $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$, the last two terms $\to 0$ by the scalar case, and $|(s_n - s)(t_n - t)| < \varepsilon^2$ eventually; so $s_n t_n \to st$.

(4) For the reciprocal, since $s_n \to s \neq 0$, eventually $|s_n| > \frac{1}{2}|s|$, so

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n|\,|s|} < \frac{2}{|s|^2}\,|s_n - s| \to 0.$$

$\blacksquare$

[A4] cf. R 3.3.

Proposition 5.2.2

Coordinatewise convergence in $\mathbb{R}^k$

For $x_n = (\alpha_{1,n}, \ldots, \alpha_{k,n})$ and $x = (\alpha_1, \ldots, \alpha_k)$ in $\mathbb{R}^k$,[3]

$$x_n \to x \iff \alpha_{j,n} \to \alpha_j \text{ for every } j = 1, \ldots, k.$$

[3] This collapses convergence in $\mathbb{R}^k$ to k separate questions about real sequences: a vector sequence converges exactly when each coordinate does, to the matching coordinate of the limit. The reason is that the norm both dominates each single coordinate gap and is dominated by their sum, so "the whole vector is close" and "every coordinate is close" come to the same thing. In practice you rarely handle $\|x_n - x\|$ directly—you check the coordinates.

Proof 5.2.2

Proof of Coordinatewise Convergence by Comparing the Norm with Its Coordinates.

(1) ($\Rightarrow$) Each $|\alpha_{j,n} - \alpha_j| \le \|x_n - x\| = \big(\sum_i |\alpha_{i,n} - \alpha_i|^2\big)^{1/2} \to 0$.

(2) ($\Leftarrow$) Given $\varepsilon > 0$, choose N_j with $|\alpha_{j,n} - \alpha_j| < \varepsilon/\sqrt{k}$ for $n \ge N_j$; with $N = \max_j N_j$,

$$\|x_n - x\| = \Big(\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2\Big)^{1/2} < \Big(k \cdot \tfrac{\varepsilon^2}{k}\Big)^{1/2} = \varepsilon.$$

■

[A4] cf. R 3.4.

The same argument works for any p-norm $\|\cdot\|_p$ or the sup norm; under the discrete metric, $p_n \to p$ iff $p_n = p$ for all but finitely many n.

5.3 Subsequences

Definition 5.3.1

Subsequence

Given a sequence $\{p_n\}$ and indices $n_1 < n_2 < n_3 < \cdots$, the sequence $\{p_{n_k}\}$ is a *subsequence* of $\{p_n\}$ (R 3.5). For instance $a_n = (-1)^n$ diverges, yet its even and odd subsequences converge, to $+1$ and -1.

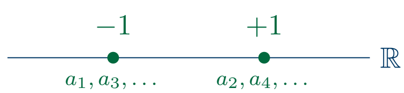

Figure 5.3.2. A DIVERGENT SEQUENCE WITH CONVERGENT SUBSEQUENCES. $a_n = (-1)^n$ never settles, but its even terms sit at $+1$ and its odd terms at -1: $a_{2k} \to 1$ and $a_{2k+1} \to -1$ (5.3.1).

Theorem 5.3.3

Convergent subsequences

1. In a compact metric space, every sequence has a convergent subsequence.
2. Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.[4]

Proof 5.3.3

Proof of the Convergent-Subsequence Theorem by Extraction at a Cluster Point.

[4] A sequence can refuse to settle and still be forced to settle *somewhere*: if its terms cannot escape—trapped in a compact space, or merely bounded in $\mathbb{R}^k$—then infinitely many of them pile up near some point, and reading those off gives a convergent subsequence. The hypothesis does all the work; drop boundedness and $p_n = n$ has no convergent subsequence at all. This is the sequential face of compactness.

(1) For (a), if $\{p_n\}$ takes only finitely many values, one value recurs infinitely often, giving a constant—hence convergent—subsequence. Otherwise the set $\{p_n\}$ is infinite, so by Bolzano–Weierstrass (4.3.3) it has a cluster point p. Choose $n_1 < n_2 < \cdots$ with $d(p_{n_k}, p) < \frac{1}{k}$ (each ball $B_{1/k}(p)$ meets the infinite value-set in terms of arbitrarily large index); then $p_{n_k} \to p$.

(2) For (b), a bounded sequence in $\mathbb{R}^k$ lies in a k-cell, which is compact (4.3.1); now apply (a).

∎

[A8] uses 4.3.3, 4.3.1; cf. R 3.6.

5.4 Cauchy Sequences and Completeness

Definition 5.4.1

Cauchy sequence

A sequence $\{p_n\}$ in (X, d) is a *Cauchy sequence* (R 3.8) if for every $\varepsilon > 0$ there is an N with $d(p_m, p_n) < \varepsilon$ for all $m, n \geq N$. No limit is named.[5]

Proposition 5.4.2

Convergent sequences are Cauchy

Every convergent sequence is Cauchy.

Proof 5.4.2

Direct Proof that Convergent Sequences Are Cauchy by Halving ε.

(1) If $p_n \to p$, given $\varepsilon > 0$ take N with $d(p_n, p) < \frac{\varepsilon}{2}$ for $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_m, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

∎

[A4] cf. R 3.11.

Definition 5.4.3

Complete metric space

(X, d) is *complete* (R 3.12) if every Cauchy sequence in X converges.[6] Thus $\mathbb{R}$, $\mathbb{R}^k$, $\mathbb{C}$, and every compact space are complete (5.5.3), while $\mathbb{Q}$ and $(0, 1)$ are not—a sequence of rationals approaching $\sqrt{2}$ is Cauchy but has no rational limit. A closed subset of a complete space is itself complete (a Cauchy sequence in it converges in the ambient space, and the limit lies in the subset since it is closed).

[5] Set this beside convergence (5.1.1): there a single target p is named and the terms must approach *it*; here only the terms appear, and they must approach *each other*. That makes "Cauchy" an internal test—checkable knowing nothing about where the sequence might be heading, or whether it heads anywhere at all. The interest of everything that follows is the gap between bunching together and actually landing.

[6] Read completeness as a promise about the *space*, not the sequence: it says the space has no holes, so any sequence whose terms bunch up (5.4.1) finds an actual limit waiting inside. The Cauchy condition is the sequence trying to converge; completeness is the space letting it. Where the promise fails—as in $\mathbb{Q}$ or $(0, 1)$ just below—the terms can bunch arbitrarily tightly around a point that simply is not there.

Theorem 5.4.4

Cauchy completion

Every metric space (X, d) embeds isometrically as a dense subspace of a complete metric space $(\widehat{X}, \widehat{d}\,)$, essentially unique, where $\widehat{X}$ is the set of Cauchy sequences in X modulo $\{p_n\} \sim \{q_n\} \iff d(p_n, q_n) \to 0$. (Adjoining the suprema of bounded sets to $\mathbb{Q}$ to build $\mathbb{R}$, Lecture 1, is this construction in disguise.)

Proof 5.4.4

Construction of the Completion by Equivalence Classes of Cauchy Sequences.

$\hookrightarrow$ Rudin, Exercise 3.24; each $p \in X$ maps to the class of the constant sequence $[p, p, \dots]$.

5.5 Completeness of Compact Spaces and of $\mathbb{R}^k$

Definition 5.5.1

Diameter and the tail of a sequence

The *diameter* of a nonempty $E \subseteq X$ is $\operatorname{diam} E = \sup_{p,q \in E} d(p, q)$ (R 3.9) (finite iff E is bounded). The *tail* of $\{p_n\}$ at N is $E_N = \{p_N, p_{N+1}, \dots\}$. Then $\{p_n\}$ is Cauchy $\iff$ $\operatorname{diam} E_N \to 0$.[7]

Proposition 5.5.2

Diameter and nested compacts

Let (X, d) be a metric space.

1. $\operatorname{diam} E = \operatorname{diam} \overline{E}$ for every nonempty $E \subseteq X$;
2. a nested sequence $K_1 \supseteq K_2 \supseteq \cdots$ of nonempty compact sets with $\operatorname{diam} K_n \to 0$ has $\bigcap_n K_n$ a single point.

Proof 5.5.2

Proof of the Diameter and Nested-Compact Claims by Approximation and Contradiction.

(1) For (1), $E \subseteq \overline{E}$ gives $\operatorname{diam} E \le \operatorname{diam} \overline{E}$. For $p, q \in \overline{E}$ and $\varepsilon > 0$, pick $p', q' \in E$ with $d(p, p'), d(q, q') \le \varepsilon$; then $d(p, q) \le d(p, p') + d(p', q') + d(q', q) \le \operatorname{diam} E + 2\varepsilon$. Taking the supremum over p, q and letting $\varepsilon \to 0$, $\operatorname{diam} \overline{E} \le \operatorname{diam} E$.

(2) For (2), $K = \bigcap_n K_n \ne \emptyset$ (4.2.2). If K held two distinct points then $\operatorname{diam} K > 0$; but $K \subseteq K_n$ gives $\operatorname{diam} K \le \operatorname{diam} K_n \to 0$, a contradiction. So K is a single point.

■

[A4+A9] uses 4.2.2; cf. R 3.10.

Theorem 5.5.3

Compact spaces and $\mathbb{R}^k$ are complete

A compact metric space is complete; and $\mathbb{R}^k$ (with the usual metric) is complete.

[7] $\operatorname{diam} E_N \le \varepsilon$ says exactly that $d(p_m, p_n) \le \varepsilon$ for all $m, n \ge N$.

Proof 5.5.3

Proof of Completeness by Trapping the Limit in Shrinking Tails.

(1) For the compact case, let $\{p_n\}$ be Cauchy in compact X. The tail-closures $\overline{E_N}$ are closed in X, hence compact (4.1.3), are nested, and $\operatorname{diam} \overline{E_N} = \operatorname{diam} E_N \to 0$ (Cauchy, 5.5.2(1)). By 5.5.2(2), $\bigcap_N \overline{E_N} = \{p\}$, and then $p_n \to p$: given $\varepsilon > 0$, $\operatorname{diam} \overline{E_N} \le \varepsilon$ for large N forces $d(p_n, p) \le \varepsilon$ there.

(2) For $\mathbb{R}^k$, a Cauchy sequence is bounded—take N with $\|p_m - p_n\| < 1$ for $m, n \ge N$, so the tail lies in $B_1(p_N)$ while the finitely many earlier terms are bounded—hence its closure is closed and bounded, so compact (Heine–Borel, 4.3.5); a Cauchy sequence in that compact set converges by the first part. So every Cauchy sequence in $\mathbb{R}^k$ converges.

■

[A7] uses 4.1.3, 4.3.5, 5.5.2; cf. R 3.11. Here $\overline{E_N}$ is compact as a closed subset of the compact X (in $\mathbb{R}^k$, closed-and-bounded).

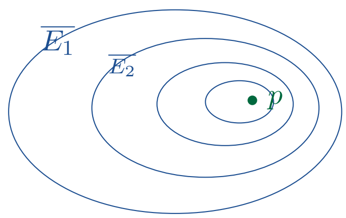

Figure 5.5.4. TRAPPING THE LIMIT IN SHRINKING TAILS. For a Cauchy sequence the tail-closures $\overline{E_1} \supseteq \overline{E_2} \supseteq \cdots$ are nested compacts with $\operatorname{diam} \overline{E_N} \to 0$; their single common point is the limit (proof of 5.5.3).

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