LECTURE 4

Compactness: Heine–Borel and Bolzano–Weierstrass

MATH S4061 | Introduction to Modern Analysis I

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Mathematics is the science of the infinite.

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CONTENTS OF THIS LECTURE

Overview. Lecture 3 defined compactness—every open cover has a finite subcover (3.4.1)—and showed it is intrinsic (3.4.4). Here we draw out its consequences: a compact set is closed and bounded, a closed subset of a compact set is compact, and nested nonempty compact sets meet. Specialising to $\mathbb{R}^k$ through nested intervals and k-cells yields the two pillars of the subject—*Heine–Borel* (in $\mathbb{R}^k$, compact = closed and bounded) and *Bolzano–Weierstrass* (every bounded infinite set has a cluster point). A short topological coda—interior, closure, connectedness—closes the chapter.

4.1 Compactness, Closedness, and Boundedness

Theorem 4.1.1

A compact set is closed and bounded

A compact subset K of a metric space (X, d) is closed and bounded (4.3.4).[1]

Proof 4.1.1

Direct Proof that the Complement Is Open.

(1) If $K = \emptyset$ it is closed and bounded vacuously, so assume $K \neq \emptyset$. Fix $p \notin K$. For each $q \in K$ set $r_q = \frac{1}{2}d(p,q) > 0$; the triangle inequality makes $B_{r_q}(p) \cap B_{r_q}(q) = \emptyset$.

(2) The balls $\{B_{r_q}(q)\}_{q \in K}$ cover K, so by compactness finitely many do: $K \subseteq B_{r_{q_1}}(q_1) \cup \cdots \cup B_{r_{q_n}}(q_n)$.

(3) Put $r = \min(r_{q_1}, \ldots, r_{q_n}) > 0$. Then $B_r(p)$ is disjoint from every $B_{r_{q_i}}(q_i)$, hence from K. So p is interior to K^c; as $p \notin K$ was arbitrary, K^c is open and K is closed.

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[Direct] ball separation; cf. R 2.34.

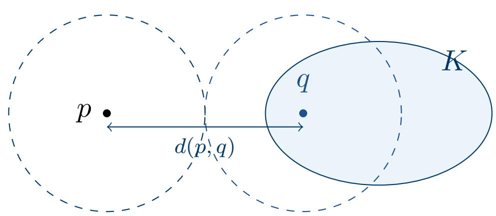

Figure 4.1.2. SEPARATING A POINT FROM A COMPACT SET. For $p \notin K$ and $q \in K$, the balls of radius $r_q = \frac{1}{2}d(p,q)$ about p and q are disjoint; as q ranges over K they cover it, and a finite subcover walls p off — so K^c is open (proof of 4.1.1).

Proposition 4.1.3

A closed subset of a compact set is compact

If K is compact and $F \subseteq K$ is closed in K, then F is compact.

Proof 4.1.3

Direct Proof that a Closed Subset of a Compact Set Is Compact.

(1) Let $\{V_\alpha\}$ be an open cover of F. Since F is closed, F^c is open in K, and $\{V_\alpha\} \cup \{F^c\}$ is an open cover of K.

(2) By compactness, finitely many cover K: $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \cup F^c$.

(3) Since $F \cap F^c = \emptyset$, dropping F^c still covers F: $F \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. So every open

[1] Bounded: fix any p; the balls $B_n(p)$ $(n \geq 1)$ cover K, so finitely many do, and $K \subseteq B_N(p)$ for the largest N.

cover of F has a finite subcover, and F is compact.

■

[Direct] cf. R 2.35.

4.2 The Finite-Intersection Property and Nested Sets

Proposition 4.2.1

The finite-intersection property

Let $\{K_\alpha\}$ be a family of nonempty compact subsets of X such that every finite subfamily has nonempty intersection. Then $\bigcap_\alpha K_\alpha \neq \emptyset$.[2]

Proof 4.2.1

Proof by Contradiction of the Finite-Intersection Property.

(1) Suppose $\bigcap_\alpha K_\alpha = \emptyset$ and fix one K_1. Then $K_1 \cap \bigcap_{\alpha\neq 1} K_\alpha = \emptyset$, i.e. $K_1 \subseteq \left(\bigcap_{\alpha\neq 1} K_\alpha\right)^c = \bigcup_{\alpha\neq 1} K_\alpha^c$.

(2) Each K_α is compact, hence closed (4.1.1), so each K_α^c is open: $\{K_\alpha^c\}_{\alpha\neq 1}$ is an open cover of K_1.

(3) By compactness of K_1, finitely many cover it: $K_1 \subseteq K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_n}^c$, i.e. $K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$ — a finite subfamily with empty intersection, contradicting the hypothesis.

■

[A9] uses 4.1.1; cf. R 2.36.

Corollary 4.2.2

Nested nonempty compact sets meet

A nested sequence $K_1 \supseteq K_2 \supseteq \cdots$ of nonempty compact sets has $\bigcap_n K_n \neq \emptyset$.[3] This is the countable nested compact corollary to Rudin's finite-intersection theorem (R 2.36).

Proposition 4.2.3

Nested closed intervals in $\mathbb{R}$

A nested sequence of nonempty closed intervals $I_n = [a_n, b_n]$ ($I_{n+1} \subseteq I_n$) has $\bigcap_n I_n = [\sup_n a_n,\ \inf_n b_n] \neq \emptyset$.

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[2] Read this as the open-cover definition turned inside out. Complementing the K_α trades "no finite subfamily covers" for "no finite subfamily has empty intersection," so a family of compact sets in which every finite batch meets cannot have empty total intersection. The moral to keep: with compactness, sharing every *finite* subfamily is enough to force one common point—nothing can leak away to infinity.

[3] A descending chain of nonempty compact sets cannot shrink away to nothing. This is the shape of almost every existence proof that follows: trap your target inside nested compact sets whose diameters tend to 0, and the single point they all share is the object you were chasing—the move behind k-cells being compact (4.3.1) and behind finding limits of sequences. Compactness is exactly what stops the chain from closing on the empty set.

Proof 4.2.3

Proof by a Supremum Argument that Nested Closed Intervals Meet.

(1) Nesting gives $a_1 \le a_2 \le \cdots \le b_2 \le b_1$, so every b_m is an upper bound for $\{a_n\}$. By the least-upper-bound property (1.6.1), $\alpha = \sup_n a_n$ exists and $a_n \le \alpha \le b_n$ for all n; thus $\alpha \in \bigcap_n I_n$.

(2) Moreover $x \in \bigcap_n I_n \iff a_n \le x \le b_n \ \forall n \iff \sup_n a_n \le x \le \inf_n b_n$, so $\bigcap_n I_n = [\sup_n a_n, \inf_n b_n]$.

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[A5] uses 1.6.1; cf. R 2.38.

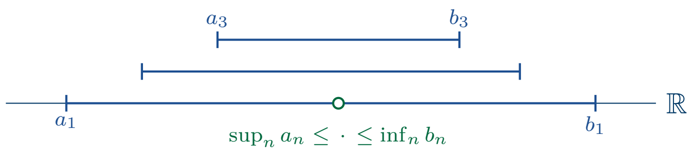

Figure 4.2.4. NESTED CLOSED INTERVALS. A descending chain $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ shares every point of $[\sup_n a_n, \inf_n b_n]$ — in particular it is never empty (4.2.3).

Definition 4.2.5

k-cell

A *k-cell* (R 2.17) is a product of closed intervals

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k] \subseteq \mathbb{R}^k.$$

Proposition 4.2.6

Nested k-cells meet

A nested sequence of nonempty k-cells $I_1 \supseteq I_2 \supseteq \cdots$ has $\bigcap_n I_n \ne \emptyset$.

Proof 4.2.6

Proof Coordinatewise that Nested k-Cells Meet.

(1) Write $I_n = [a_1^{(n)}, b_1^{(n)}] \times \cdots \times [a_k^{(n)}, b_k^{(n)}]$. For each fixed coordinate i the intervals $[a_i^{(n)}, b_i^{(n)}]$ are nested, so by 4.2.3 some $\alpha_i \in \bigcap_n [a_i^{(n)}, b_i^{(n)}]$.

(2) Then $\alpha = (\alpha_1, \ldots, \alpha_k) \in \bigcap_n I_n$.

■

[A5] applies 4.2.3 in each coordinate; cf. R 2.39.

4.3 Heine–Borel and Bolzano–Weierstrass

Theorem 4.3.1

Every k-cell is compact

Every k-cell $I \subseteq \mathbb{R}^k$ is compact.[4]

Proof 4.3.1

Proof by Repeated Bisection that the k-cell Is Compact.

(1) Let $\delta = \left(\sum_{i=1}^{k}(b_i - a_i)^2\right)^{1/2}$, so $d(x,y) \le \delta$ for all $x, y \in I$. Suppose I is *not* compact: some open cover $\{G_\alpha\}$ has no finite subcover.

(2) Bisecting each edge at its midpoint splits I into 2^k sub-cells. If each had a finite subcover, so would I; hence some sub-cell I_1 has none. Iterating gives nested cells $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$, each without a finite subcover, with $\operatorname{diam} I_n \le 2^{-n}\delta$ (each bisection halves every edge, hence the diagonal).

(3) By 4.2.6 some $x \in \bigcap_n I_n$. As $\{G_\alpha\}$ covers I, $x \in G_{\alpha_0}$ for some α_0, and G_{α_0} open gives $r > 0$ with $B_r(x) \subseteq G_{\alpha_0}$. Choose N with $2^{-N}\delta < r$ (Archimedean, 1.7.1); then $I_N \subseteq B_r(x) \subseteq G_{\alpha_0}$ — so I_N is covered by the single G_{α_0}, contradicting that it has no finite subcover.

■

[A7] bisection and nested cells (4.2.6); cf. R 2.40.

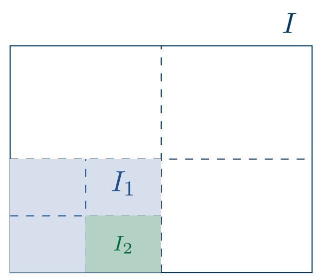

Figure 4.3.2. Bisecting a k-cell. A k-cell without a finite subcover has a 2^k-bisection inheriting the defect; iterating gives nested cells $I \supset I_1 \supset I_2$ of diameter $\to 0$, trapping a point that a single cover element must contain (proof of 4.3.1).

Theorem 4.3.3

Bolzano–Weierstrass

Every infinite subset E of a compact metric space[5] K has a cluster point in K.

Proof 4.3.3

Proof by Contradiction that E Would Be Finite.

[4] Why is a solid box compact when an open or unbounded set is not? Bisect it into 2^k smaller boxes: if no finite part of a cover handled the whole box, some sub-box inherits the same defect, and repeating traps a single point whose own little neighbourhood—sitting inside *one* set of the cover—then swallows an entire tiny box, the contradiction. The trap works because halving sends the diameter to 0 across finitely many coordinates; that is just what breaks down in infinitely many dimensions, and why Heine–Borel (4.3.5) is a fact about $\mathbb{R}^k$ and not metric spaces at large.

[5] Rudin reserves the name *Bolzano–Weierstrass* (R 2.42) for the bounded-$\mathbb{R}^k$ case; the general compact statement here is R 2.37, whence the $\mathbb{R}^k$ case follows by enclosing a bounded set in a k-cell (4.3.1).

(1) Suppose no point of K is a cluster point of E. Then each $q \in K$ has $r_q > 0$ with $B_{r_q}(q) \cap E \subseteq \{q\}$.

(2) The balls $\{B_{r_q}(q)\}_{q \in K}$ cover K; by compactness finitely many do. Each holds at most one point of E, so E is finite — contradicting that E is infinite.

■

[A9] cf. R 2.37.

Definition 4.3.4

Bounded set

$E \subseteq X$ is *bounded* (R 2.18) if $E \subseteq B_r(p)$ for some $p \in X$ and $r > 0$.

Theorem 4.3.5

Heine–Borel

For $E \subseteq \mathbb{R}^k$ the following are equivalent:[6]

1. E is closed and bounded;
2. E is compact;
3. every infinite subset of E has a cluster point in E.

Proof 4.3.5

Proof of the Heine–Borel Theorem by Cycling (a)⇒(b)⇒(c)⇒(a).

(1) (a)⇒(b). Bounded E lies in some k-cell I; I is compact (4.3.1) and $E \subseteq I$ is closed, so E is compact (4.1.3).

(2) (b)⇒(c). This is Bolzano–Weierstrass (4.3.3).

(3) (c)⇒(a), contrapositive. If E is *not closed*, a cluster point $q \notin E$ exists; choose $p_n \in E$ with $\|p_n - q\| < \frac{1}{n}$. The set $\{p_n\}$ is infinite (a finite range would repeat some value $p \in E$ infinitely, forcing $q = p \in E$), and its only possible cluster point is $q \notin E$: any $y \in E$ has $\|p_n - y\| \geq \|q - y\| - \frac{1}{n} \geq \frac{1}{2}\|q - y\|$ eventually. If E is *not bounded*, choose $p_n \in E$ with $\|p_n\| > n$; then $\{p_n\}$ is infinite (the norms are unbounded) with no cluster point, since any $B_1(y)$ holds only finitely many p_n. Either way (c) fails.

■

[A9+A4] uses 4.1.3, 4.3.1, 4.3.3; cf. R 2.41.

4.4 Interior, Closure, and Connectedness

Definition 4.4.1

Interior and closure

[6] The equivalence (a) $\iff$ (b) is *special to* $\mathbb{R}^k$: in a general metric space a compact set is closed and bounded (4.1.1), but closed-and-bounded need not be compact.

For $E \subseteq X$, the *interior* $\mathring{E}$ is the set of interior points of E (2.3.3); it is open, and E is open iff $E = \mathring{E}$. The *closure* (R 2.27) is

$$\overline{E} = E \cup \{\text{cluster points of } E\} \ (3.2.1);$$

it is closed — indeed the smallest closed set containing E.[7]

Definition 4.4.2

Connected and disconnected

A metric space (X, d) is *connected* (R 2.45) if its only subsets that are both open and closed ("clopen") are $\emptyset$ and X. Equivalently, X is *disconnected* when $X = A \cup B$ for nonempty, disjoint, open sets A, B.[8]

Theorem 4.4.3

Connected subsets of $\mathbb{R}$ are exactly the intervals

A subset $E \subseteq \mathbb{R}$ is connected (4.4.2) if and only if it is an *interval* (R 2.47): whenever $x, y \in E$ and $x < z < y$, we have $z \in E$.[9]

Proof 4.4.3

Proof That a Subset of $\mathbb{R}$ Is Connected Iff It Is an Interval.

(1) ($\Rightarrow$), contrapositive. Suppose E is *not* an interval: there are $x, y \in E$ and $z \notin E$ with $x < z < y$. In the subspace E put

$$U = (-\infty, z) \cap E, \qquad V = (z, \infty) \cap E.$$

Both are open in E (intersections of E with open subsets of $\mathbb{R}$), nonempty ($x \in U$, $y \in V$), and disjoint. Since $z \notin E$, we have $U \cup V = E$. So $E = U \cup V$ is a disconnection, and E is not connected (4.4.2).

(2) ($\Leftarrow$). Suppose E has the interval property but is *not* connected, say $E = A \cup B$ with A, B nonempty, open in E, and disjoint. Pick $x \in A$ and $y \in B$; relabel so that $x < y$, and set

$$z = \sup\,(A \cap [x, y]),$$

which exists by the least-upper-bound property (1.6.1) since $A \cap [x, y]$ is nonempty

[7]If $C \supseteq E$ is closed then C contains every cluster point of E, so $C \supseteq \overline{E}$; and $\overline{E}$ is itself closed, so it is the least such set.

[8]The picture is just "one piece." A clopen set other than $\emptyset$ or X would split X into two parts, each open, with neither approaching the other—a clean break. Requiring the only clopen sets to be $\emptyset$ and X says precisely that no such split exists, so the space hangs together in a single piece.

[9]The two directions split the labour cleanly. "Not an interval" means E has a visible gap at some z, and the two open rays on either side of z are the separation you want—no metric subtlety needed. The reverse direction is the only place the completeness of $\mathbb{R}$ enters: a disconnection of an interval would have to have a boundary point $z = \sup A$, and the least-upper-bound property is exactly what produces that z and traps it between the two halves. This is the fact our Intermediate Value Theorem (7.5.5) rests on—a continuous image of an interval is connected, hence again an interval.

(x lies in it) and bounded above by y. As $x \leq z \leq y$ and E has the interval property, $z \in E$, so $z \in A$ or $z \in B$.

(3) If $z \in A$, then $z \neq y$ (as $y \in B$), so $z < y$. Since A is open in E, some $(z-\varepsilon, z+\varepsilon) \cap E \subseteq A$; points just above z then lie in $A \cap [x, y]$, contradicting $z = \sup(A \cap [x, y])$.

(4) If $z \in B$, then $z \neq x$ (as $x \in A$), so $x < z$. Since B is open in E, some $(z - \varepsilon, z + \varepsilon) \cap E \subseteq B$; in particular $(z - \varepsilon, z) \cap E \subseteq B$ is disjoint from A, so no point of $A \cap [x, y]$ exceeds $z - \varepsilon$ — again contradicting $z = \sup(A \cap [x, y])$. Either way we reach a contradiction, so E is connected.

■

[A4+A5] uses 1.6.1; cf. R 2.47.

Definition 4.4.4

Dense and perfect

Let $E \subseteq X$. We call E *dense* in X (R 2.18) if $\overline{E} = X$ — equivalently, if every nonempty open subset of X meets E.[10] We call E *perfect* (R 2.43) if E is closed and every point of E is a cluster point of E — that is, $E = E'$, where E' is the derived set of cluster points.

Example 4.4.5

$\mathbb{Q}$ is dense in $\mathbb{R}$

$\mathbb{Q}$ is dense in $\mathbb{R}$: by the density of the rationals (1.7.2), every open interval $(a, b) \subseteq \mathbb{R}$ contains a rational. Hence every real number is a cluster point of $\mathbb{Q}$, and $\overline{\mathbb{Q}} = \mathbb{R}$.

[10] Density says E leaves no room: every point of X is either in E or arbitrarily well approximated by it, so you can never carve out an open patch of X that avoids E entirely. The two phrasings are the same fact seen from the closure (4.4.1) and from its complement.

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