LECTURE 3

The Topology of Metric Spaces: Open and Closed Sets, and Compactness

MATH S4061 | Introduction to Modern Analysis I

Tuesday, 2 June 2026

> *Topology is precisely the mathematical discipline that allows the passage from local to global.*
>
> René Thom

Contents of this lecture

3.1 The Algebra of Open Sets

3.2 Limit Points and Closed Sets

3.3 The Subspace Topology

3.4 Compactness

Overview. Lecture 2 introduced metric spaces and the open ball (2.3.1). This lecture builds the *topology* they generate: which sets are open—closed under arbitrary unions and *finite* intersections—then the limit (cluster) points that define *closed* sets, the clean duality E open $\iff$ E^c closed, how openness is *relative* to a subspace, and finally *compactness*: the open-cover/finite-subcover property that, unlike open or closed, is *intrinsic* to a set. One method recurs throughout: to show a set is open, write it as a union of balls centred at its own points. The companion text is Rudin, Chapter 2, especially R 2.18–R 2.33.

3.1 The Algebra of Open Sets

Definition 3.1.1

Indexed unions and intersections

Let A be an index set and $V_\alpha \subseteq X$ for each $\alpha \in A$. The *union* and *intersection* are

$$\bigcup_{\alpha\in A} V_\alpha = \{x : x \in V_\alpha \text{ for some } \alpha\}, \qquad \bigcap_{\alpha\in A} V_\alpha = \{x : x \in V_\alpha \text{ for every } \alpha\};$$

in particular $p \in \bigcap_\alpha V_\alpha \iff p \in V_\alpha$ for all α. This is the indexed-family notation of R 2.9.

Proposition 3.1.2

Unions and finite intersections of open sets

Let (X, d) be a metric space with $V_\alpha \subseteq X$ open for every $\alpha \in A$.

1. $\bigcup_{\alpha\in A} V_\alpha$ is open (any union of open sets is open);
2. $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n}$ is open (any *finite* intersection is open).[1]

Proof 3.1.2

Direct Proof that Unions and Finite Intersections of Open Sets Are Open.

(1) For the union, let $p \in \bigcup_\alpha V_\alpha$, say $p \in V_\beta$. As V_β is open, some $B_{r_p}(p) \subseteq V_\beta \subseteq \bigcup_\alpha V_\alpha$; so p is interior and the union is open.

(2) For a finite intersection, let $p \in V_{\alpha_1} \cap \cdots \cap V_{\alpha_n}$. Each V_{α_i} open gives $r_i > 0$ with $B_{r_i}(p) \subseteq V_{\alpha_i}$. Put $r = \min(r_1, \ldots, r_n) > 0$; then $B_r(p) \subseteq V_{\alpha_i}$ for every i, so $B_r(p) \subseteq \bigcap_i V_{\alpha_i}$. The minimum of *finitely* many positive numbers is positive—this is where finiteness is used.

■

[Direct] uses 2.3.3; finiteness enters in (2); cf. R 2.24.

Example 3.1.3

Infinite intersections can fail

The bound $r = \min r_i$ becomes an infimum that may vanish. In $(\mathbb{R}, |\cdot|)$ the open intervals $I_n = (-\frac{1}{n}, \frac{1}{n})$ satisfy

$$\bigcap_{n\geq 1} I_n = \{0\},$$

which is not open in $\mathbb{R}$. So "finite" in 3.1.2(2) cannot be dropped (cf. R 2.24).

Proposition 3.1.4

X and $\emptyset$ are open

In any metric space (X, d), both X and $\emptyset$ are open in X (cf. R 2.24).

[1] The asymmetry is the whole point. Openness gives every point a ball of room; for a *union* a point needs that room from only the one piece it already lies in, so any number of open sets stays open. An *intersection* demands room from *all* the pieces at once—and finitely many positive radii have a positive minimum, while infinitely many can shrink to zero and leave no room at all (3.1.3). Finiteness is exactly what keeps that minimum positive.

Proof 3.1.4

Direct Proof that X and $\emptyset$ Are Open, the Second Case Vacuously.

(1) For any $p \in X$ and any $r > 0$, $B_r(p) \subseteq X$, so every point of X is interior: X is open.

(2) $\emptyset$ has no points, so "every point is interior" holds vacuously: $\emptyset$ is open.

■

[Direct]

Proposition 3.1.5

Open sets are exactly unions of open balls

A set $V \subseteq X$ is open if and only if it is a union of open balls. This is the union-of-neighbourhoods viewpoint behind R 2.19 and R 2.24.

Proof 3.1.5

Constructive Proof that Open Sets Are Exactly Unions of Open Balls.

(1) If V is open, each $p \in V$ has an $r_p > 0$ with $B_{r_p}(p) \subseteq V$. Every ball lies in V, and each p lies in its own ball, so

$$V = \bigcup_{p \in V} B_{r_p}(p),$$

a union of open balls.

(2) Conversely any union of open balls is open: each ball is open (2.3.5), and a union of open sets is open (3.1.2).

■

[Constr] balls are open by 2.3.5; the converse uses 3.1.2.

3.2 Limit Points and Closed Sets

Definition 3.2.1

Cluster (limit) point

Let (X, d) be a metric space and $E \subseteq X$. A point $p \in X$ is a *cluster point*[2] of E if every open ball $B_r(p)$ contains a point of E other than p: for all $r > 0$ there is $q \in E$ with $q \in B_r(p)$ and $q \neq p$. Consequently a finite set has no cluster points.

Definition 3.2.2

Isolated point

[2] Also called a *limit point* (Rudin's term) or *accumulation point*. Because every ball holds a point of $E \setminus \{p\}$, it in fact holds infinitely many points of E: a ball with only finitely many would have a least positive distance from p among them, so a smaller ball meets $E \setminus \{p\}$ nowhere (cf. R 2.20).

A point $p \in E$ is an *isolated point* of E if some ball meets E only at p: there is $r > 0$ with $B_r(p) \cap E = \{p\}$ (R 2.18).

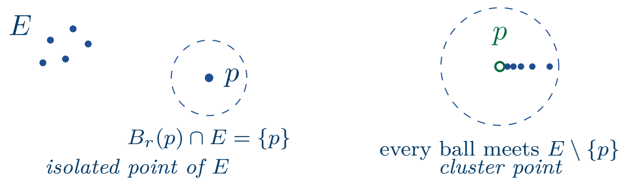


Figure 3.2.3. ISOLATED VERSUS CLUSTER POINTS. Left: p is *isolated* — a ball meets E only at p. Right: p is a *cluster point* — every ball, however small, meets E away from p (3.2.1, 3.2.2).

Definition 3.2.4

Closed set

E is *closed in* X if it contains all of its cluster points: whenever $p \in X$ is a cluster point of E, $p \in E$.[3] To prove E is *not* closed, exhibit one cluster point of E lying outside E (R 2.18).

Example 3.2.5

Cluster points and closedness

In $(\mathbb{R}, |\cdot|)$:

- $E = \{\frac{1}{n} : n \geq 1\}$ has 0 as its only cluster point[4] (every $B_r(0)$ contains some $\frac{1}{n}$, by the Archimedean property, 1.7.1); since $0 \notin E$, E is *not* closed, whereas $\{0\} \cup \{\frac{1}{n}\}$ *is*.
- $\mathbb{Z}$ is closed because it has *no* cluster points: each integer is isolated, and any $p \notin \mathbb{Z}$ has $\operatorname{dist}(p, \mathbb{Z}) > 0$, so a small ball about p misses $\mathbb{Z}$. Vacuously, $\mathbb{Z}$ holds all its cluster points.
- Every real is a cluster point of $\mathbb{Q}$ (by density, 1.7.2); $\mathbb{Q}$ is far from closed.

Compare Rudin's limit-point and closed-set definitions in R 2.18 and the nearby limit-point facts in R 2.20.

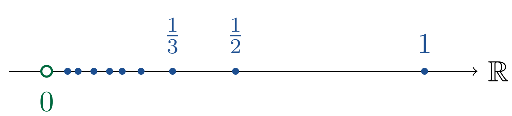


Figure 3.2.6. THE CLUSTER POINT OF $\{1/n\}$. The points $1/n$ bunch arbitrarily close to 0, so every ball about 0 meets $E = \{1/n\}$; yet $0 \notin E$, so E is not closed (3.2.5).

Remark 3.2.7

Open, closed, neither, both

[3] "Closed" is not "not open"; it means closed *under taking limits*. Read it as a sealing condition: anything E gets arbitrarily close to—a cluster point—already lies in E, so no limiting process can leak out of the set. That is what it means for a closed set to have no missing edges, and it is why closed sets are exactly those containing the limits of their own convergent sequences.

[4] Each $\frac{1}{n}$ is isolated, and any $p \notin \{0\} \cup E$ has a neighbourhood missing E; so 0 is the sole cluster point.

"Closed" is relative: say "E is closed *in* X." And open and closed are not opposites—in $(\mathbb{R}, |\cdot|)$ a set may be open, closed, neither, or both (here $p < q$). The interval (p, q) is open; $[p, q]$ is closed; $(p, q]$ is neither; $\mathbb{Q}$ is neither; and $\emptyset$ and X are *both* ("clopen"). Compare the complement duality and set algebra in R 2.23–R 2.24.

Theorem 3.2.8

E open $\iff$ E^c closed

For $E \subseteq X$ with complement $E^c = \{p \in X : p \notin E\}$: E is open in X if and only if E^c is closed in X.[5]

Proof 3.2.8

Proof via Cluster Points that E Is Open $\iff$ E^c Is Closed.

(1) ($\Rightarrow$) Let E be open and $p \in E$. Some $B_r(p) \subseteq E$, so $B_r(p) \cap E^c = \emptyset$ and p is *not* a cluster point of E^c. Thus no point of E is a cluster point of E^c: every cluster point of E^c lies in E^c, i.e. E^c is closed.

(2) ($\Leftarrow$) Let E^c be closed and $p \in E$. Then $p \notin E^c$; since E^c holds all its cluster points, p is not one, so some $B_r(p) \cap E^c = \emptyset$, i.e. $B_r(p) \subseteq E$. Hence p is interior and E is open.

■

[Direct] cf. R 2.23; recall $(E^c)^c = E$.

Lemma 3.2.9

De Morgan

For any family $\{E_\alpha\}_{\alpha \in A}$ of subsets of X,

$$\Big(\bigcup_\alpha E_\alpha\Big)^c = \bigcap_\alpha E_\alpha^c, \qquad \Big(\bigcap_\alpha E_\alpha\Big)^c = \bigcup_\alpha E_\alpha^c.$$

Proof 3.2.9

Proof of De Morgan's Laws by Element-Chasing.

(1) For the first identity,

$$p \in \Big(\bigcup_\alpha E_\alpha\Big)^c \iff p \notin \bigcup_\alpha E_\alpha \iff p \notin E_\alpha \ \forall \alpha \iff p \in E_\alpha^c \ \forall \alpha \iff p \in \bigcap_\alpha E_\alpha^c.$$

(2) The second follows by replacing each E_α with E_α^c and taking complements.

■

[Direct] cf. R 2.22.

Proposition 3.2.10

Intersections and finite unions of closed sets

[5] Treat this less as a fact than as a tool: it lets any statement about open sets be reread as one about closed sets, and back, just by complementing. That is how 3.2.10 falls out of 3.1.2 for free—complement, apply the open version, complement again—so the closed-set theorems never need a fresh proof.

Let $F_\alpha \subseteq X$ be closed in X for every $\alpha \in A$.

1. $\bigcap_{\alpha \in A} F_\alpha$ is closed (any intersection of closed sets is closed);
2. $F_{\alpha_1} \cup \cdots \cup F_{\alpha_n}$ is closed (any *finite* union is closed).

Proof 3.2.10

Proof by Dualising the Open Case that Intersections and Finite Unions of Closed Sets Are Closed.

(1) For the intersection, de Morgan gives $\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha F_\alpha^c$. Each F_α^c is open (3.2.8), and a union of open sets is open (3.1.2); so the complement is open and $\bigcap_\alpha F_\alpha$ is closed (3.2.8).

(2) For a finite union, $\left(F_{\alpha_1} \cup \cdots \cup F_{\alpha_n}\right)^c = F_{\alpha_1}^c \cap \cdots \cap F_{\alpha_n}^c$ is a *finite* intersection of open sets, hence open (3.1.2); so the union is closed. (Infinite unions can fail, dually to 3.1.3.)

■

[Direct] via 3.2.8, 3.2.9, 3.1.2; cf. R 2.24. A proof straight from the cluster-point definition (the board's exercise) also works.

3.3 The Subspace Topology

Proposition 3.3.1

Relative openness

Let (X, d) be a metric space, $Y \subseteq X$ a subspace, and $E \subseteq Y$. Then E is open in Y if and only if $E = Y \cap G$ for some G open in X. (Here the ball in Y is $B_r^Y(p) = \{x \in Y : d(x,p) < r\} = B_r(p) \cap Y$.)

Proof 3.3.1

Proof of Relative Openness by Carving Y-balls from X-balls.

(1) ($\Leftarrow$) Suppose $E = Y \cap G$, G open in X. For $p \in E$ we have $p \in G$, so some $B_{r_p}(p) \subseteq G$; intersecting with Y, $B_{r_p}^Y(p) = B_{r_p}(p) \cap Y \subseteq G \cap Y = E$. Thus p is interior in Y, and E is open in Y.

(2) ($\Rightarrow$) Suppose E is open in Y: each $p \in E$ has $r_p > 0$ with $B_{r_p}^Y(p) \subseteq E$. Put $G = \bigcup_{p \in E} B_{r_p}(p)$, open in X as a union of X-balls (3.1.5). Then

$$G \cap Y = \bigcup_{p \in E} \left(B_{r_p}(p) \cap Y\right) = \bigcup_{p \in E} B_{r_p}^Y(p) = E.$$

■

[Direct] cf. R 2.30.

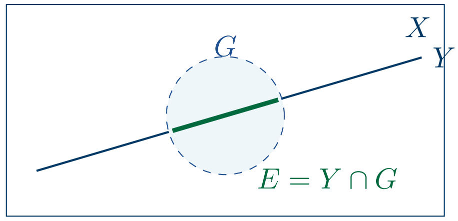


Figure 3.3.2. Open in a subspace. E is open in Y exactly when $E = Y \cap G$ for some G open in the whole space X: an X-ball G meets the subspace Y in the relative set E (3.3.1).

Remark 3.3.3

Openness by balls at its own points

The recurring method, used in 3.1.5 and 3.3.1: to prove a set is open, write it as a union of open balls centred at its own points (cf. R 2.19, R 2.30).

3.4 Compactness

Definition 3.4.1

Open cover; compact set

Let (X, d) be a metric space and $K \subseteq X$. An *open cover* of K is a family $\{V_\alpha\}$ of sets open in X with $K \subseteq \bigcup_\alpha V_\alpha$. The set K is *compact* if every open cover of K admits a *finite subcover*: indices $\alpha_1, \dots, \alpha_n$ with $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$.[6] This combines Rudin's open-cover definition R 2.31 and compactness definition R 2.32.

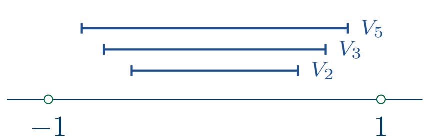
Example 3.4.2

Two non-compact sets

In $(\mathbb{R}, |\cdot|)$:

- $\mathbb{R}$ is not compact: the cover $V_n = (-n, n)$ $(n \geq 1)$ has $\bigcup_n V_n = \mathbb{R}$, but any finite subfamily lies in V_N with $N = \max n_i$, missing all $|x| \geq N$.
- $(-1, 1)$ is not compact: $V_n = (-1 + \frac{1}{n}, 1 - \frac{1}{n})$ $(n \geq 2)$ cover $(-1, 1)$, yet any finite subfamily is contained in the largest V_N, missing points near ± 1.

These are direct open-cover failures for Rudin's compactness definition R 2.32; compare the Heine–Borel theorem R 2.41.



[6] The naive candidate "closed and bounded" depends on the ambient space and the metric; the open-cover definition is *intrinsic* (3.4.4) and survives passage to continuous images, which is why it is the right notion.

Figure 3.4.3. AN OPEN COVER WITH NO FINITE SUBCOVER. The intervals $V_n = (-1 + \frac{1}{n}, 1 - \frac{1}{n})$ cover $(-1, 1)$, but each stops short of ± 1; a finite subfamily reaches only the largest V_N — so $(-1, 1)$ is not compact (3.4.2).

Proposition 3.4.4

Compactness is intrinsic

Let $K \subseteq Y \subseteq X$, where (X, d) is a metric space and Y carries the subspace metric $d|_{Y \times Y}$. Then K is compact in Y if and only if K is compact in X. So one may say simply "K is compact," with no ambient space.

Proof 3.4.4

Proof that Compactness Is Intrinsic by Trading Y-covers for X-covers.

(1) ($\Leftarrow$) Assume K compact in X. Let $\{V_\alpha\}$ be a Y-open cover of K. By 3.3.1 each $V_\alpha = G_\alpha \cap Y$ with G_α open in X, and $\{G_\alpha\}$ covers K. By X-compactness, $K \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$; intersecting with Y (and using $K \subseteq Y$), $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$.

(2) ($\Rightarrow$) Assume K compact in Y. Let $\{G_\alpha\}$ be an X-open cover of K. Set $V_\alpha = G_\alpha \cap Y$, open in Y (3.3.1) and covering K. By Y-compactness, $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$.

■

[Direct] uses 3.3.1; cf. R 2.33.

Index of Objects

3.1.1 Definition: Indexed unions and intersections 2

3.1.2 Proposition: Unions and finite intersections of open sets 2

3.1.4 Proposition: X and $\emptyset$ are open 2

3.1.5 Proposition: Open sets are exactly unions of open balls 3

3.2.1 Definition: Cluster (limit) point 3

3.2.2 Definition: Isolated point 3

3.2.4 Definition: Closed set 4

3.2.8 Theorem: E open $\iff$ E^c closed 5

3.2.9 Lemma: De Morgan 5

3.2.10 Proposition: Intersections and finite unions of closed sets 5

3.3.1 Proposition: Relative openness 6

3.4.1 Definition: Open cover; compact set 7

3.4.4 Proposition: Compactness is intrinsic 8