LECTURE 1

From the Naturals to the Reals: Induction, Ordered Fields, and Completeness

MATH S4061 | Introduction to Modern Analysis I

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God made the integers; all else is the work of man.

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CONTENTS OF THIS LECTURE

Overview. The course opens with a deceptively simple question—is $0.999\ldots = 1$?—whose honest answer requires building the real numbers from the ground up. We begin with the natural numbers and induction, construct $\mathbb{Z}$ and $\mathbb{Q}$, and isolate the field and order axioms they satisfy. The rationals then fall short for analysis: $\sqrt{2}$ is missing, and the sets bounding it have neither a largest nor a smallest element. Repairing that gap is the *least-upper-bound property*, which characterises $\mathbb{R}$; from it follow the Archimedean property and the density of $\mathbb{Q}$—and, at last, the resolution of $0.999\ldots = 1$. The companion text is Rudin, Chapter 1, especially the ordered-field and least-upper-bound material R 1.5–R 1.20.

1.1 The Natural Numbers and Induction

Definition 1.1.1

The natural numbers

The *natural numbers* are $\mathbb{N} = \{0, 1, 2, 3, \dots\}$[1], generated from 0 by the *successor function*

$$S : \mathbb{N} \to \mathbb{N}, \qquad n \mapsto n + 1.$$

Axiom 1.1.2

Induction

If $A \subseteq \mathbb{N}$ satisfies $0 \in A$ and $n \in A \Rightarrow n + 1 \in A$, then $A = \mathbb{N}$. Rudin treats induction as prerequisite proof background rather than a numbered analysis result.

Theorem 1.1.3

The principle of mathematical induction

Let $P(n)$ be a statement for each $n \in \mathbb{N}$. If $P(0)$ holds, and $P(n) \Rightarrow P(n + 1)$ for every n, then $P(n)$ holds for all $n \in \mathbb{N}$. This is the proof principle behind later uses of induction; Rudin assumes it as background.

Proof 1.1.3

Proof of the Induction Principle from the Axiom.

(1) Let $A = \{n \in \mathbb{N} : P(n) \text{ holds}\}$.

(2) Since $P(0)$ holds, $0 \in A$; and $P(n) \Rightarrow P(n + 1)$ says $n \in A \Rightarrow n + 1 \in A$.

(3) By the induction axiom (1.1.2), $A = \mathbb{N}$; that is, $P(n)$ holds for every n.

■

[Direct] uses 1.1.2.

Example 1.1.4

Gauss's sum

For every integer $n \geq 1$, $\quad 1 + 2 + \cdots + n = \dfrac{n(n + 1)}{2}$. This is included as induction practice rather than as a Rudin-numbered item.

Proof 1.1.4

Proof by Induction of Gauss's Formula.

(1) For $n = 1$ the formula reads $1 = \dfrac{1 \cdot 2}{2}$, which holds.

[1] We take $0 \in \mathbb{N}$; Rudin instead starts the positive integers at 1. The convention is harmless—only the base of an induction shifts (Gauss's sum, 1.1.4, begins at $n = 1$).

(2) Assume $1 + \cdots + n = \frac{n(n+1)}{2}$. Adding $n+1$ to both sides,

$$1 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2},$$

which is the formula for $n+1$. The result follows by induction (1.1.3). ■

[A10] from the base $n = 1$ (apply 1.1.3 to the shifted statement).

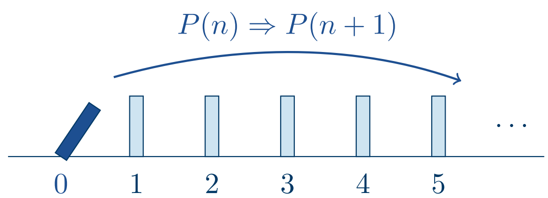

Figure 1.1.5. Induction as toppling dominoes. The base case $P(0)$ topples the first domino; the inductive step $P(n) \Rightarrow P(n+1)$ passes the fall to the next, so every $P(n)$ falls in turn—the conclusion of 1.1.2 and 1.1.3.

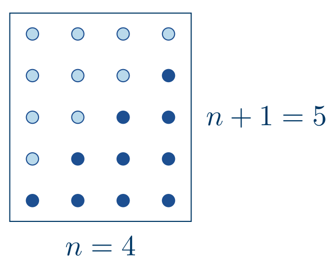

Figure 1.1.6. Gauss's sum as half a rectangle. Two copies of the staircase $1+2+3+4$ interlock into a 4×5 grid, so $2(1+2+3+4) = 4 \cdot 5$; in general $1 + \cdots + n = \frac{n(n+1)}{2}$ (1.1.4).

1.2 The Integers and the Rationals

Construction 1.2.1

The integers

The *integers* are signed naturals with the two zeros glued together:

$$\mathbb{Z} := (\{+,-\} \times \mathbb{N})/\sim, \qquad (+,0) \sim (-,0).$$

Writing $+n, -n$ for the classes recovers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Rudin treats $\mathbb{Z}$ as background and begins the analysis notation with the rational field in R 1.4.

Construction 1.2.2

The rationals

The *rationals* are equivalence classes of integer pairs with nonzero second coordinate:

$$\mathbb{Q} := \{(p,q) \in \mathbb{Z} \times \mathbb{Z} : q \neq 0\}/\!\sim, \qquad (p,q) \sim (r,s) \iff ps = rq.$$

The class of (p,q) is written $\dfrac{p}{q}$. Rudin records the rational-number notation and convention in R 1.4.

Remark 1.2.3

One number, many fractions

A rational *is* the whole class, e.g. $\left[\frac{1}{2}\right] = \left\{\frac{1}{2}, \frac{2}{4}, \frac{-1}{-2}, \frac{17}{34}, \dots\right\}$. In practice one names it by its *preferred representative*: the irreducible fraction $\dfrac{a}{b}$ with $\gcd(a,b) = 1$ and the sign carried in the numerator. This is the equivalence-class bookkeeping behind the notation in R 1.4.

1.3 Fields and Ordered Fields

Definition 1.3.1

Field

A *field* is a set F with two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ satisfying eleven axioms[2]—making F an abelian group under $+$, making $F \setminus \{0\}$ an abelian group under $\cdot$, with multiplication distributing over addition.

Definition 1.3.2

Order

An *order* on a set S is a relation $<$ such that (cf. R 1.5)

(1) (*trichotomy*) for $x, y \in S$, exactly one of $x < y$, $y < x$, $x = y$ holds;

(2) (*transitivity*) $x < y$ and $y < z$ imply $x < z$.

A set carrying an order is *totally* (or *linearly*) ordered. Rudin calls this an ordered set in R 1.6.

Definition 1.3.3

Ordered field

An *ordered field* is a field F with an order $<$ compatible with the operations (R 1.17):

(1) $y < z \Rightarrow x + y < x + z$ for all x;

(2) $x > 0$ and $y > 0 \Rightarrow xy > 0$.

Remark 1.3.4

The rationals are an ordered field

[2] Five for addition (A1–A5), five for multiplication (M1–M5), and the distributive law (D); the full statements are R 1.12.

$\mathbb{Q}$ is an ordered field. The rest of the lecture asks what it *lacks*—and why that makes it unfit for analysis. Compare Rudin's ordered-field definition R 1.17 and the real-field existence theorem R 1.19.

1.4 Why the Rationals Are Not Enough

Proposition 1.4.1

No rational squares to 2

There is no $x \in \mathbb{Q}$ with $x^2 = 2$ (R 1.1).

Proof 1.4.1

Proof by Contradiction of the Irrationality of $\sqrt{2}$.

(1) Suppose $x = \dfrac{p}{q} \in \mathbb{Q}$ is irreducible with $x^2 = 2$. Then $p^2 = 2q^2$, so p^2 is even, hence p is even (an odd integer has an odd square); write $p = 2k$.

(2) Substituting, $4k^2 = 2q^2$, so $q^2 = 2k^2$ is even, hence q is even.

(3) Then $2 \mid \gcd(p, q)$, contradicting that $\dfrac{p}{q}$ is irreducible. So no such x exists.

∎

[A9] cf. R 1.1; uses "n^2 even $\Rightarrow n$ even".

Proposition 1.4.2

The rationals have a gap

Let $A = \{p \in \mathbb{Q} : p > 0,\ p^2 < 2\}$ and $B = \{p \in \mathbb{Q} : p > 0,\ p^2 > 2\}$. Then A has no largest element and B has no smallest element.

Proof 1.4.2

Proof by an Explicit Construction of the Gap.

(1) Given $p > 0$, set

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} > 0.$$

(2) A computation gives

$$q^2 - 2 = \frac{2\,(p^2 - 2)}{(p + 2)^2},$$

so $q^2 - 2$ has the same sign as $p^2 - 2$: if $p \in A$ then $q \in A$, and if $p \in B$ then $q \in B$.

(3) Also $q - p = -\dfrac{p^2 - 2}{p + 2}$, so $q > p$ when $p \in A$ and $q < p$ when $p \in B$.

(4) Hence every $p \in A$ is exceeded by a larger $q \in A$, so A has no largest element; and every $p \in B$ is undercut by a smaller $q \in B$, so B has no smallest element.

∎

[Constr] cf. R 1.1.

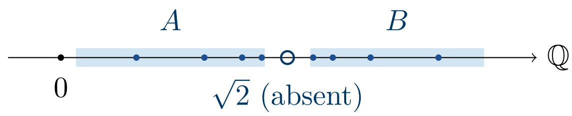

Figure 1.4.3. THE GAP AT $\sqrt{2}$. The positive rationals A ($p^2 < 2$) climb toward $\sqrt{2}$ with no largest member; B ($p^2 > 2$) descend toward it with no smallest; in $\mathbb{Q}$ the meeting point $\sqrt{2}$ is simply absent.

1.5 Bounds and the Supremum

Definition 1.5.1

Upper bound

Let S be an ordered set and $E \subseteq S$. An element $\beta \in S$ is an *upper bound* for E if $x \leq \beta$ for every $x \in E$ (R 1.7).

Definition 1.5.2

Least upper bound (supremum)

β is the *least upper bound* of E if it is an upper bound for E and no smaller element is: $\alpha < \beta$ implies α is not an upper bound. We write $\beta = \sup E = \operatorname{lub} E$ (R 1.8).

Definition 1.5.3

Lower bound and infimum

Dually, $\beta \in S$ is a *lower bound* for E if $\beta \leq x$ for every $x \in E$, and the *greatest lower bound* (or *infimum*) if no larger element is a lower bound: $\beta < \alpha$ implies α is not a lower bound. We write $\beta = \inf E = \operatorname{glb} E$ (cf. R 1.8). Since $\inf E = -\sup(-E)$, the least-upper-bound property (1.6.1) carries a matching greatest-lower-bound property.

Remark 1.5.4

The ε-characterisation of the supremum

The supremum, when it exists, is *unique* (by trichotomy, 1.3.2) and *need not* belong to E.[3] The least-upper-bound condition has a working reformulation that is the tool behind nearly every supremum argument in the course:

$$\alpha = \sup E \quad \Longleftrightarrow \quad \alpha \text{ is an upper bound of } E, \text{ and for every } \varepsilon > 0 \text{ there is } x \in E \text{ with } x > \alpha - \varepsilon.$$

The first clause says α is an upper bound; the second says nothing smaller is—any proposed reduction $\alpha - \varepsilon$ is already overshot by a member of E. Dually,

$$\alpha = \inf E \quad \Longleftrightarrow \quad \alpha \text{ is a lower bound of } E, \text{ and for every } \varepsilon > 0 \text{ there is } x \in E \text{ with } x < \alpha + \varepsilon.$$

This rephrasing (cf. R 1.8) is what one actually reaches for: to use a supremum, feed it an ε and extract the witness x.

[3] For $E = \{x \in \mathbb{Q} : 0 < x^2 < 2\}$ the supremum is $\sqrt{2} \notin E$ (indeed $\notin \mathbb{Q}$); for $E = \{1 - \frac{1}{n} : n \geq 1\}$ it is $1 \notin E$. When $\sup E$ *does* lie in E it is the *maximum* of E.

Proof 1.5.4

Proof of the ε-Characterisation.

(1) ($\Rightarrow$) Let $\alpha = \sup E$. Then α is an upper bound. Given $\varepsilon > 0$, the smaller number $\alpha - \varepsilon$ is *not* an upper bound (else α would not be the *least* one), so some $x \in E$ exceeds it: $x > \alpha - \varepsilon$.

(2) ($\Leftarrow$) Suppose α is an upper bound with the stated ε-property, and let $\gamma < \alpha$. Put $\varepsilon = \alpha - \gamma > 0$; the property yields $x \in E$ with $x > \alpha - \varepsilon = \gamma$, so γ is not an upper bound. Hence no $\gamma < \alpha$ bounds E, and $\alpha = \sup E$.

■

[Direct] uses 1.5.2; the inf case is dual.

1.6 The Real Number System

Theorem 1.6.1

The real number system

There is an ordered field $\mathbb{R} \supseteq \mathbb{Q}$ in which every nonempty subset that is bounded above has a least upper bound. It is unique up to isomorphism.[4] This is Rudin's existence theorem for the real field (R 1.19).

Remark 1.6.2

Completeness and density

The property in 1.6.1 is *completeness*: $\mathbb{R}$ is complete iff it has the least-upper-bound property. A first consequence is that $\mathbb{Q}$ is *dense* in $\mathbb{R}$—between any two reals lies a rational (1.7.2; cf. R 1.19–R 1.20).

Remark 1.6.3

Three ways to build $\mathbb{R}$

- *Accept it*—take 1.6.1 on faith and learn to use sup.
- *Dedekind cuts* (1.6.4)—realise each real as a downward-closed set of rationals.
- *Completion*—the general procedure that completes any metric space (the abstract construction is deferred). Here it realises $\mathbb{R}$ via *Cauchy sequences of rationals*: a sequence in $\mathbb{Q}$ that "ought to converge" *is* declared a real number. The missing $\sqrt{2}$ reappears as, for instance, the recursion

$$q_0 = 1, \qquad q_{n+1} = \frac{2q_n + 2}{q_n + 2} \qquad \text{(terms } 1,\ 1.4,\ 1.41,\ \dots\text{)},$$

whose map $p \mapsto \dfrac{2p+2}{p+2}$ is exactly the gap-construction of 1.4.2: it fixes the value $\sqrt{2}$ and drives every starting rational toward it, so $q_n \to \sqrt{2}$. Many different rational sequences crowd toward the same target, so one declares any two that "converge together" *equivalent*, and a real number is such an equivalence class.

[4] An isomorphism of ordered fields is a bijection preserving $+$, $\cdot$, and $<$; "unique up to isomorphism" means any two complete ordered fields are indistinguishable as such. Two were built in 1872: Dedekind's cuts (*Stetigkeit und irrationale Zahlen*) and Cantor's equivalence classes of Cauchy sequences of rationals.

Rudin states the existence theorem abstractly in R 1.19; these are model-building viewpoints.

Definition 1.6.4

Dedekind cut

A *cut* is a set $\alpha \subseteq \mathbb{Q}$ such that

(1) $\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$;

(2) α is downward closed: $r \in \alpha$ and $s < r$ imply $s \in \alpha$;

(3) α has no largest element.

The cuts, suitably ordered, *are* the real numbers; this is one construction behind Rudin's real-field theorem R 1.19.

Example 1.6.5

$\sqrt{2}$ *as a cut*

The real number $\sqrt{2}$ is the cut

$$\alpha = \{q \in \mathbb{Q} : q < 0\} \ \cup\ \{q \in \mathbb{Q} : q^2 < 2\}.$$

This is the set A of 1.4.2 together with the non-positive rationals; the former "gap" is now a bona fide element (cf. R 1.1, R 1.19).

1.7 The Archimedean Property, Density, and Decimals

Theorem 1.7.1

Archimedean property

If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n with $nx > y$.[5] This is the Archimedean part of R 1.20.

Proof 1.7.1

Proof by Contradiction of the Archimedean Property.

(1) Suppose not: the set $A = \{nx : n \in \mathbb{N}\}$ is bounded above. By the least-upper-bound property (1.6.1) it has a supremum $\alpha = \sup A$.

(2) Since $x > 0$, $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound of A: some $m \in \mathbb{N}$ has $mx > \alpha - x$.

(3) Then $(m+1)x > \alpha$, yet $(m+1)x \in A$—contradicting $\alpha = \sup A$. So A is not bounded above, and in particular some $nx > y$.

∎

[A5+A9] cf. R 1.20; via the supremum; uses 1.6.1.

[5]Named by Otto Stolz in the 1880s after Archimedes, who postulated it in *On the Sphere and the Cylinder*; the idea is older still—Euclid records it (*Elements* V, Def. 4) and credits Eudoxus.

Theorem 1.7.2

Density of $\mathbb{Q}$ in $\mathbb{R}$

If $x, y \in \mathbb{R}$ and $x < y$, then there is a rational q with $x < q < y$. This is the density part of R 1.20.

Proof 1.7.2

Proof by the Archimedean Property of the Density of $\mathbb{Q}$.

(1) Since $y - x > 0$, the Archimedean property (1.7.1) gives $n \in \mathbb{N}$ with $n(y-x) > 1$, i.e. $ny - nx > 1$.

(2) Since $ny - nx > 1$, the interval (nx, ny) contains an integer m (§1.A.1).

(3) Dividing by $n > 0$, $x < \dfrac{m}{n} < y$, so $q = \dfrac{m}{n}$ works.

∎

[Direct] cf. R 1.20; uses 1.7.1.

Construction 1.7.3

Real numbers as decimals

Every *nonnegative* real is the supremum of its finite decimal truncations (a negative real is then named by its sign):

$$1.4112\ldots \longmapsto \sup\{1,\ 1.4,\ 1.41,\ 1.411,\ \ldots\}.$$

This uses the least-upper-bound and Archimedean machinery in R 1.19–R 1.20.

Example 1.7.4

$0.999\ldots = 1$

This settles the opening question. Put $x = 1 - 0.999\ldots$; then $0 \le x < 10^{-k}$ for every $k \in \mathbb{N}$. By the Archimedean property (1.7.1)[6] no positive real lies below every 10^{-k}, so $x = 0$—that is, $0.999\ldots = 1$. More generally every terminating decimal equals its repeating-9 twin ($2.5 = 2.4999\ldots$, and so on), so

$$\mathbb{R} = \{\text{decimal expansions}\}/\!\sim,$$

the quotient that identifies each terminating decimal with that twin (cf. R 1.20).

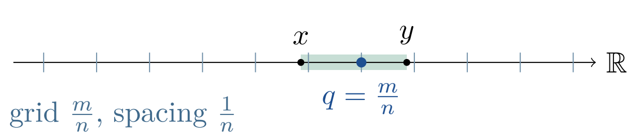

Figure 1.7.5. Density of $\mathbb{Q}$ in $\mathbb{R}$. Choosing n with $1/n < y - x$ makes the grid of multiples m/n finer than the gap, so some m/n lands strictly between x and y (1.7.2).

[6] If $x > 0$, the Archimedean property gives n with $nx > 1$; taking k with $10^k \ge n$ then yields $10^k x > 1$, i.e. $x > 10^{-k}$.

Appendix

Supplementary material—additional proofs, context, and other treatments; not part of the lecture proper.

Supplement 1.A.1

An interval longer than 1 contains an integer

If $s, t \in \mathbb{R}$ with $t - s > 1$, then there is an integer m with $s < m < t$. (Used silently in the density proof, 1.7.2.)

Proof 1.A.1

Direct Proof from the Archimedean Property.

(1) The integers exceeding s are nonempty (by the Archimedean property) and bounded below by s, so they have a least element m; thus $m - 1 \le s < m$.

(2) Hence $m \le s + 1 < t$, and therefore $s < m < t$.

■

[Direct] uses 1.7.1; cf. R 1.20.

Supplement 1.A.2

Every nonnegative real has a decimal expansion

Construction 1.7.3 names each nonnegative real as the supremum of its truncations; here we build those truncations digit by digit and confirm the supremum is the number we began with. Given $x \ge 0$, there are an integer $a_0 \ge 0$ and digits $d_1, d_2, \dots \in \{0, \dots, 9\}$ whose truncations

$$t_k \;=\; a_0 + \sum_{i=1}^{k} \frac{d_i}{10^i} \qquad \text{satisfy} \qquad 0 \le x - t_k < 10^{-k},$$

so that $x = \sup_k t_k$. The rule below always takes the *largest* admissible digit, so a number with a terminating expansion receives it ($1 = 1.000\ldots$), and the repeating-9 twin (1.7.4) is the single expansion this rule never produces. The key order input is R 1.20.

Proof 1.A.2

Direct Proof by Greedy Digit Selection.

(1) For the integer part, let a_0 be the greatest integer with $a_0 \le x$ — it exists by the least-integer argument of 1.A.1, applied to the integers exceeding x, and $a_0 \ge 0$ since $x \ge 0$. Then $t_0 = a_0$ obeys $0 \le x - t_0 < 1$.

(2) For the digits, suppose t_{k-1} has been built with $0 \le x - t_{k-1} < 10^{-(k-1)}$. Let d_k be the largest digit in $\{0, \dots, 9\}$ with $t_{k-1} + d_k\, 10^{-k} \le x$; the digit 0 qualifies, and $d_k \le 9$ because $x < t_{k-1} + 10 \cdot 10^{-k} = t_{k-1} + 10^{-(k-1)}$. Put $t_k = t_{k-1} + d_k\, 10^{-k}$. Then $t_k \le x$, while maximality of d_k gives $x < t_k + 10^{-k}$; hence $0 \le x - t_k < 10^{-k}$ and the invariant persists.

(3) For the supremum: every $t_k \le x$, so x is an upper bound. If $\beta < x$, then $x - \beta > 10^{-k}$ for some k (Archimedean property, 1.7.1), so $t_k > x - 10^{-k} > \beta$; thus no

$\beta < x$ bounds the truncations, and $x = \sup_k t_k$.

■

[Direct] uses 1.7.1, 1.A.1.

Index of Objects